ALGEBRAIC CYCLES ON ABELIAN VARIETIES

Brief description of my AWS 2024 lectures and projects

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My lectures will be about Chow groups of abelian varieties. There is a wealth of material here! In view of the limited available time, I have chosen to focus in my lectures on results that can be formulated—though not necessarily proven—without using the language of Chow motives. (One of the projects goes in that direction, though.) Here is a summary of what I have in mind:

Lectures 1 and 2. I will start with a brief reminder about Chow groups and the available structures on them. If X is an abelian variety over a field k, its Chow group $\mathrm{CH}(X)$ is a graded ring in two different ways: using the intersection product, with grading by codimension of cycles, and via Pontryagin product, with grading by the dimension of cycles. If X^t is the dual abelian variety and $\mathscr{P}$ is the Poincaré bundle on $X \times X^t$, the class $\mathrm{ch}(\mathscr{P}) \in \mathrm{CH}(X \times X^t)_{\mathbb{Q}}$ is a correspondence that defines a transformation

$$\mathscr{F} \colon \mathrm{CH}(X)_{\mathbb{Q}} \to \mathrm{CH}(X^t)_{\mathbb{Q}}\,,$$

called the Fourier transform. The first main result that I will discuss is that $\mathscr{F}$ is an isomorphism that exchanges (up to signs) the two ring structures.

The Fourier tranform is not compatible with the gradings by dimension or codimension of cycles. This leads to a second main result. Namely, if for an integer s we define

$$\mathrm{CH}^i_{(s)}(X) = \left\{\alpha \in \mathrm{CH}^i(X)_{\mathbb{Q}} \;\middle|\; \mathscr{F}(\alpha) \in \mathrm{CH}^{g-i+s}(X^t)_{\mathbb{Q}}\right\}$$

then it turns out that (with $g = \dim(X)$)

$$\mathrm{CH}^i(X)_{\mathbb{Q}} = \bigoplus_{s=i-g}^{i} \mathrm{CH}^i_{(s)}(X)\,,$$

and $\mathrm{CH}^i_{(s)}(X)$ is precisely the subspace of $\mathrm{CH}^i(X)_{\mathbb{Q}}$ of elements α with the property that $[n]_X^*(\alpha) = n^{2i-s} \cdot \alpha$ for all $n \in \mathbb{Z}$. This is known as the Beauville decomposition.

As I shall explain, there is a nice and insightful picture of $\mathrm{CH}(X)$ that we can draw, in which the symmetry given by Fourier duality has a prominent role. I hope to be able to state all main results in the first lecture; in the second lecture I will then focus on the key ideas in the proofs.

Literature: The main results discussed in the first two lectures are due to Mukai [16] (on the level of the derived category, which we will not discuss) and Beauville [2], [3].

Lecture 3. In the third lecture I will first explain that there is yet more structure on $\mathrm{CH}(X)_{\mathbb{Q}}$. Namely, the Lie algebra $\mathfrak{sl}_{2,\mathbb{Q}}$ acts on $\mathrm{CH}(X)_{\mathbb{Q}}$, and this gives rise to a decomposition of $\mathrm{CH}(X)_{\mathbb{Q}}$ as a direct sum of finite dimensional $\mathfrak{sl}_{2,\mathbb{Q}}$-representations. After this, we shall take a closer look at 0-cycles, which form a sub-ring $\mathrm{CH}_0(X) \subset \mathrm{CH}(X)$ with respect to the Pontryagin product (denoted by $\star$). The $\star$-powers of the ideal $I = \mathrm{Ker}\big(\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}\big)$ give us a filtration of $\mathrm{CH}_0(X)$. We shall discuss a result of Roĭtman that, over an algebraically closed base field, $I^{\star r}$ is a $\mathbb{Q}$-vector space for all $r \geq 2$. Further, we shall discuss that the Beauville decomposition gives a natural splitting of this filtration. An immediate consequence of this is that $I^{\star(g+1)} = 0$.

Literature: The $\mathfrak{sl}_2$-action on $\mathrm{CH}(X)_{\mathbb{Q}}$ can in fact be "integrated" to an action of the algebraic group SL_2. The idea that such an action exists has its roots in the work of Mukai. The fact that we have a Lie algebra action (even on the level of Chow motives) was proven by Künnemann in [13], though he does not present the result as a Lie algebra action. The SL_2-action was studied in Polishchuk's PhD thesis [21]. A very nice exposition is given in [4]. What we shall discuss about 0-cycles on abelian varieties can be found in [2], [3] and [6].

Lecture 4. In the last lecture, I will discuss some results that give an indication of the "size" of Chow groups. Firstly, over a field $k \subseteq \overline{\mathbb{F}}_p$ we have that $\mathrm{CH}(X)_{\mathbb{Q}} = \oplus_i \mathrm{CH}^i_{(0)}(X)$, and conjecturally the ℓ-adic cycle class maps $\mathrm{cl}_\ell\colon \mathrm{CH}^i_{(0)}(X) \otimes \mathbb{Q}_\ell \to H^{2i}\big(X_{\bar{k}}, \mathbb{Q}_\ell(i)\big)$ (for $\ell \neq p$) are injective; in this case we conclude that the Chow groups are actually rather small, and for 0-cycles it follows that $I^{\star 2} = 0$. By contrast, if the base field is algebraically closed and uncountable, the Chow groups are very big, in a sense that we will make precise. For 0-cycles, the bound $I^{\star(g+1)} = 0$ is sharp, and by using the $\mathfrak{sl}_{2,\mathbb{Q}}$-action on $\mathrm{CH}(X)_{\mathbb{Q}}$ we shall be able to draw conclusions about other Beauville summands $\mathrm{CH}^i_{(s)}(X)$ as well. If time permits, I will at the end briefly discuss some important conjectures about the existence of a natural filtration on Chow groups and how, in the case of abelian varieties, this relates to the Beauville decomposition.

Literature: The results about the case where $k \subseteq \overline{\mathbb{F}}_p$ can be found in [13], Section 7, which uses ideas from [22]. The idea that, in general, Chow groups are very big, is a deep insight of Mumford [17]. To read about this, I recommend [23], Chapter 10.

Reading material. Detailed lecture notes will be made available. For students who already want to read more about these topics, Beauville's papers [2], [3] as well as Bloch's paper [6] are a very good starting point. As Chow groups (algebraic cycles modulo rational equivalence) play a key role in my lectures, it is helpful if students have already seen those prior to the AWS. Fulton's book [10] is the canonical reference for intersection theory, but most of this book goes far beyond what we shall need, so I recommend to use this only as a reference work. In fact, it should suffice if you have read Appendix A from Hartshorne's book [12]. Another very good starting point is [23], Chapter 9. If you are interested in motivic aspects, I can recommend for a general introduction the book by Murre, Nagel and Peters [18] or (more panoramic) the book by André [1]. To see the motivic theory in action, in the setting of my lectures, I recommend to look at the important paper [7] by Deninger and Murre.

There are two projects that I should like to propose.

Project 1: Integral aspects of Fourier duality. The main drawback of the usual Fourier transform is that it requires $\mathbb{Q}$-coefficients, so that we lose all information about torsion classes. In this project we want to study if we can construct an integral version of Fourier duality, or at least a version that works with $\mathbb{Z}[\frac{1}{n}]$-coefficients, for a suitable number n. Part of the project is to explore what exactly we would mean by a Fourier duality with (almost) integral coefficients.

An integral version of Fourier duality "up to a factor N" is given in [2], Proposition $3'$, but this result is not effective and does not give a concrete value for N that works. We can try to make this effective by combining the argument of Beauville with the integral version of Grothendieck–Riemann–Roch proven by Pappas in [20]. (The result of Pappas works over fields of characteristic 0, though what we need is also valid in characteristic p.)

A different approach to integral aspects of Fourier duality is given in [15]. This has recently found a nice application in [5].

A second part of this project is to explore integral versions of the Beauville decomposition. The leading question is whether we can find (given some abelian variety X) some "small" subring $\Lambda \subset \mathbb{Q}$ such that we already have a Beauville decomposition of $\mathrm{CH}(X) \otimes \Lambda$. In the presence of torsion, the conditions stated in [3], Proposition 1 are no longer all equivalent; but if we have a Fourier duality with coefficients in the ring Λ, we could still use it to obtain a Beauville decomposition.

Project 2: A refined Beauville decomposition for non-simple abelian varieties. Suppose we have an abelian variety X which decomposes as a product of smaller abelian varieties, say $X = X_1 \times \cdots \times X_t$. For $\underline{n} = (n_1, \ldots, n_t) \in \mathbb{Z}^t$ we have the endomorphism $[\underline{n}]_X$ of X given by $(x_1, \ldots, x_t) \mapsto (n_1 x_1, \ldots, n_t x_t)$. The usual multiplication-by-n maps are the special case where $n_1 = \cdots = n_t = n$. The goal of this project is to investigate if we can find a decomposition of $\mathrm{CH}(X)$ into simultaneous "eigenspaces" for all operators $[\underline{n}]_X^*$.

This project is an ideal opportunity to learn something about Chow motives. It turns out that the problem can be studied in a clean way using only some basic principles of the theory of motives, and the book-keeping that is involved becomes much more transparent.

We can vary on this theme. A nice case to study is when X is a supersingular abelian variety, say over an algebraically closed field k of characteristic p. Fakhruddin [9] has shown that the Beauville decomposition is in this case essentially trivial: only the summands $\mathrm{CH}^i_{(s)}(X)$ with $s \in \{0, 1\}$ may be nonzero. This uses the important geometric fact, due to Oort [19], that in this case X is isogenous to E^g for a supersingular elliptic curve E. Hence X has a very large endomorphism algebra, namely $\mathrm{End}^0(X) \cong M_g(D)$, where D is a quaternion algebra over $\mathbb{Q}$. The group of units of this algebra acts on $\mathrm{CH}(X)$ and on the Chow motive of X, and one can try to relate the arguments of Fakhruddin to this action, using the results of [14].

References

[1] Y. André, Une introduction aux motifs (motifs purs, motifs mixtes, périodes). Panoramas et Synthèses 17. Soc. Math. de France, 2004.

[2] A. Beauville, Quelques remarques sur la transformation de Fourier dans l'anneau de Chow d'une variété abélienne. In: Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math. 1016, pp. 238–260.

[3] A. Beauville, Sur l'anneau de Chow d'une variété abélienne. Math. Ann. 273 (1986), no. 4, 647–651.

[4] A. Beauville, The action of SL_2 on abelian varieties. J. Ramanujan Math. Soc. 25 (2010), no. 3, 253–263.

[5] T. Beckmann, O. de Gaay Fortman, Integral Fourier transforms and the integral Hodge conjecture for one-cycles on abelian varieties. Compos. Math. 159 (2023), no. 6, 1188–1213.

[6] S. Bloch, Some elementary theorems about algebraic cycles on Abelian varieties. Invent. Math. 37 (1976), no. 3, 215–228.

[7] C. Deninger, J. Murre, Motivic decomposition of abelian schemes and the Fourier transform. J. reine angew. Math. 422 (1991), 201–219.

[8] D. Eisenbud, J. Harris, 3264 and all that—a second course in algebraic geometry. Cambridge University Press, Cambridge, 2016.

[9] N. Fakhruddin, On the Chow groups of supersingular varieties. Canad. Math. Bull. 45 (2002), no. 2, 204–212.

[10] W. Fulton, Intersection theory. Ergebnisse der Math. und ihrer Grenzgebiete (3. Folge), 2. Springer-Verlag, 1998.

[11] W. Fulton, Introduction to intersection theory in algebraic geometry. CBMS Regional Conference Series in Math. 54. American Math. Soc., 1984.

[12] R. Hartshorne, Algebraic Geometry. Graduate Texts in Math. 52. Springer-Verlag, 1977.

[13] K. Künnemann, A Lefschetz decomposition for Chow motives of abelian schemes. Invent. Math. 113 (1993), no. 1, 85–102.

[14] B. Moonen, On the Chow motive of an abelian scheme with non-trivial endomorphisms. J. reine angew. Math. 711 (2016), 75–109.

[15] B. Moonen, A. Polishchuk, Divided powers in Chow rings and integral Fourier transforms. Adv. Math. 224 (2010), no. 5, 2216–2236.

[16] S. Mukai, Duality between $\mathrm{D}(X)$ and $\mathrm{D}(\hat{X})$ with its application to Picard sheaves. Nagoya Math. J. 81 (1981), 153–175.

[17] D. Mumford, Rational equivalence of 0-cycles on surfaces. J. Math. Kyoto Univ. 9 (1968), 195–204.

[18] J. Murre, J. Nagel, C. Peters, Lectures on the theory of pure motives. Univ. Lecture Series 61. American Mathematical Society, 2013.

[19] F. Oort, Subvarieties of moduli spaces. Invent. Math. 24 (1974), 95–119.

[20] G. Pappas, Integral Grothendieck–Riemann–Roch theorem. Invent. Math. 170 (2007), no. 3, 455–481.

[21] A. Polishchuk, Biextensions, Weil representations on derived categories and theta-functions. PhD Thesis, Harvard University, 1996.

[22] C. Soulé, Groupes de Chow et K-théorie de variétés sur un corps fini. Math. Ann. 268 (1984), no. 3, 317–345.

[23] C. Voisin, Hodge theory and complex algebraic geometry. II. (Translated from the French by L. Schneps.) Cambridge Studies in Adv. Math. 77. Cambridge University Press, 2003.

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