In complete

families

over

im perfect

fields

a

Perceived funniness is higher in incomplete and imperfect [puns].

2.1

$$\begin{array}{l} M_g \\ \Big\downarrow \text{Torelli} \\ A_g \end{array}$$

M_g moduli space smooth curves genus g

Not complete

Deligne - Mumford

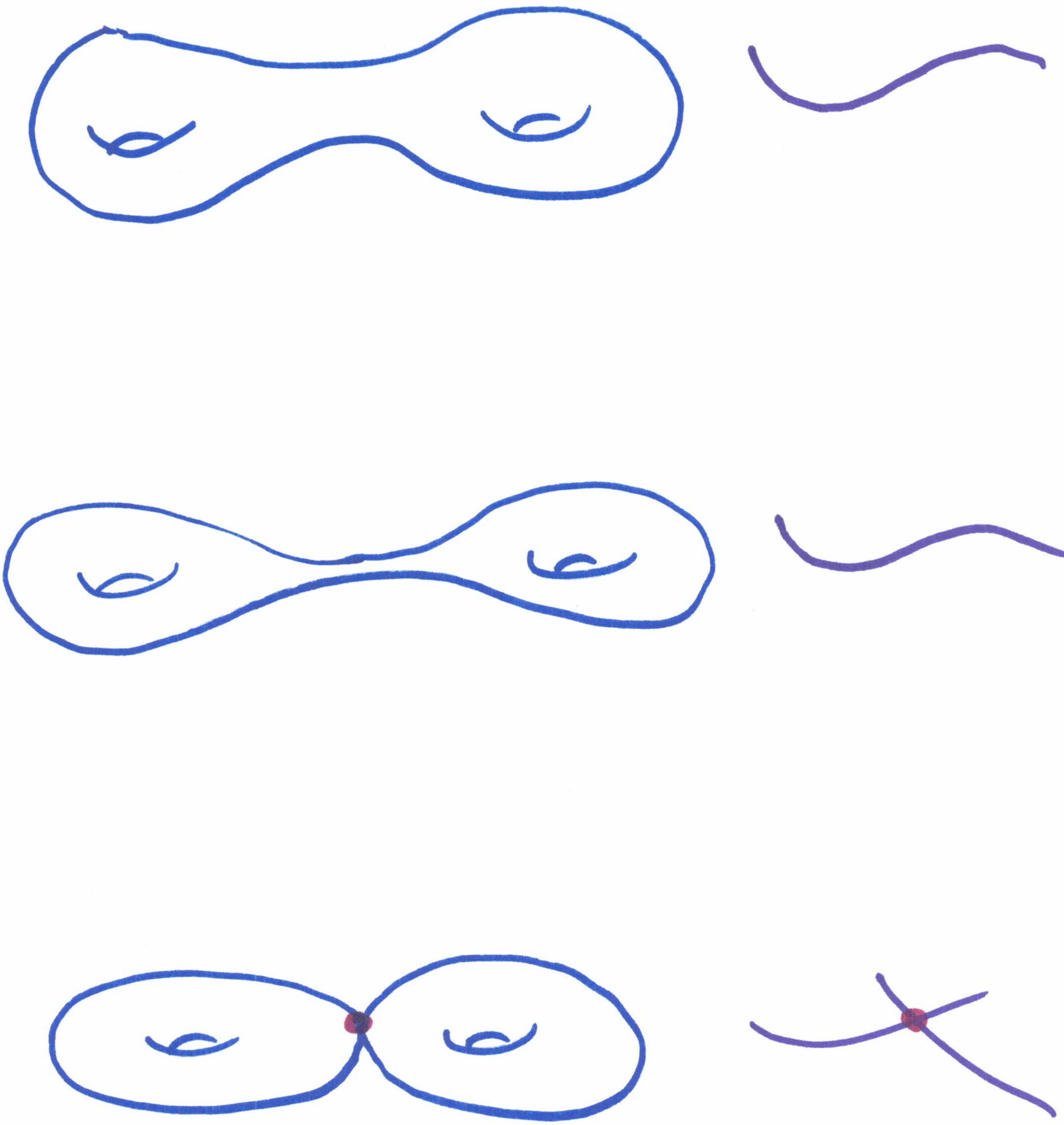
$\overline{M_g}$

A_g moduli space p.p. abelian varieties dim g

Not complete

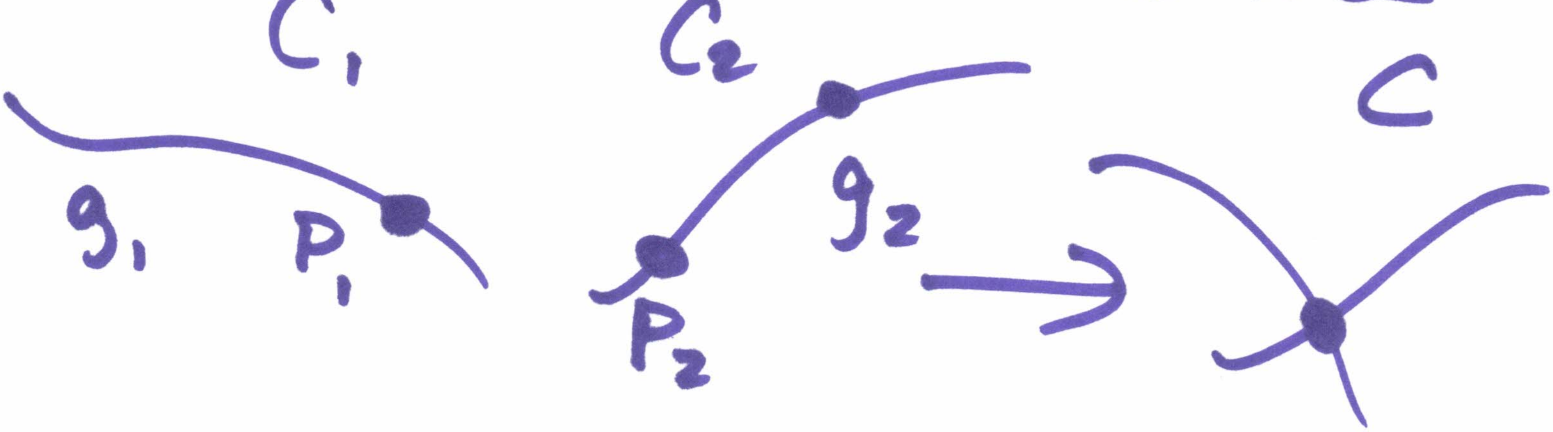
$\widehat{A_g}$

2.1



2.2 Singular curves compact type

dual graph is a tree



$$\kappa : \overline{M}_{g_1,1} \times \overline{M}_{g_2,1} \longrightarrow \overline{M}_{g_1+g_2}$$

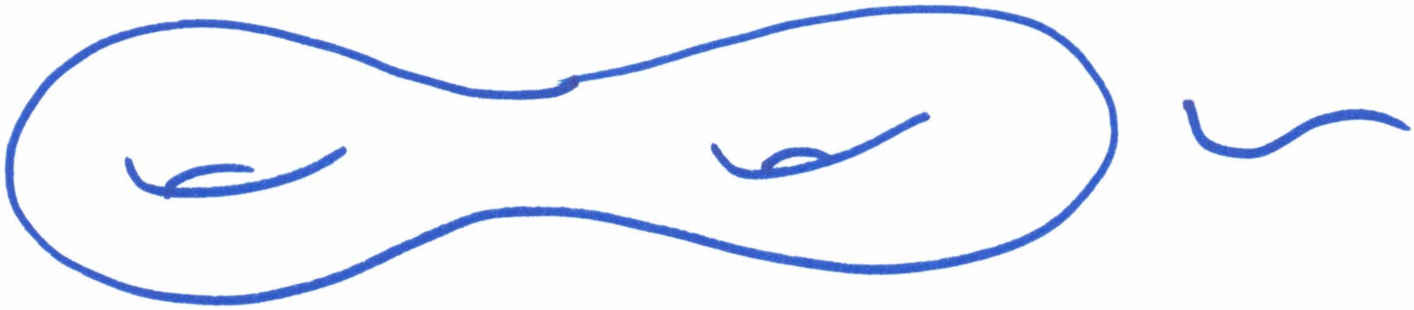
Image called

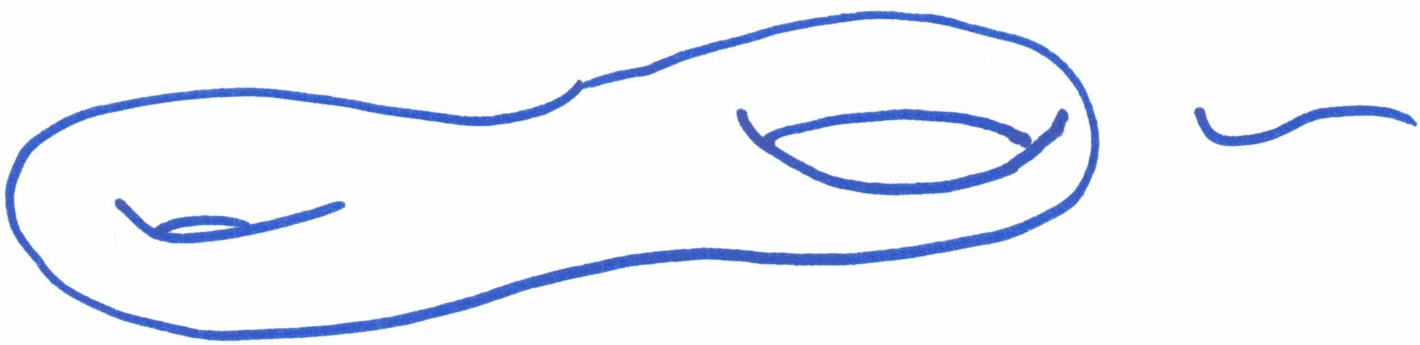
boundary divisor $\quad \Delta_{g_1} = \Delta_{g_2}$

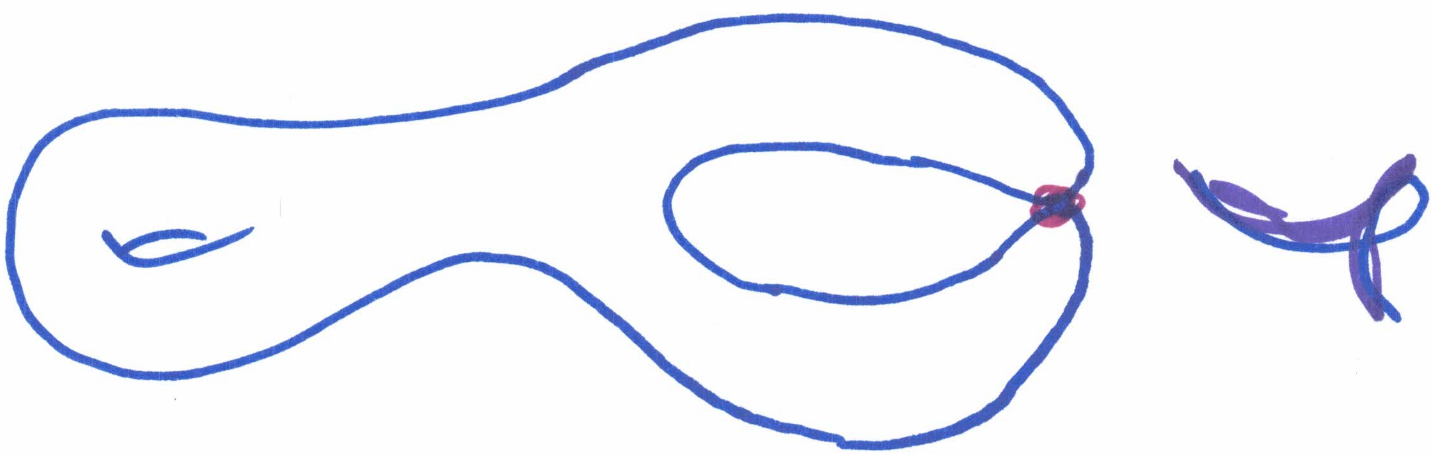
$$\mathrm{Jac}(C) = \mathrm{Jac}(C_1) \oplus \mathrm{Jac}(C_2)$$

$$\Delta_{g_1} = \overline{M}_g - M_g$$

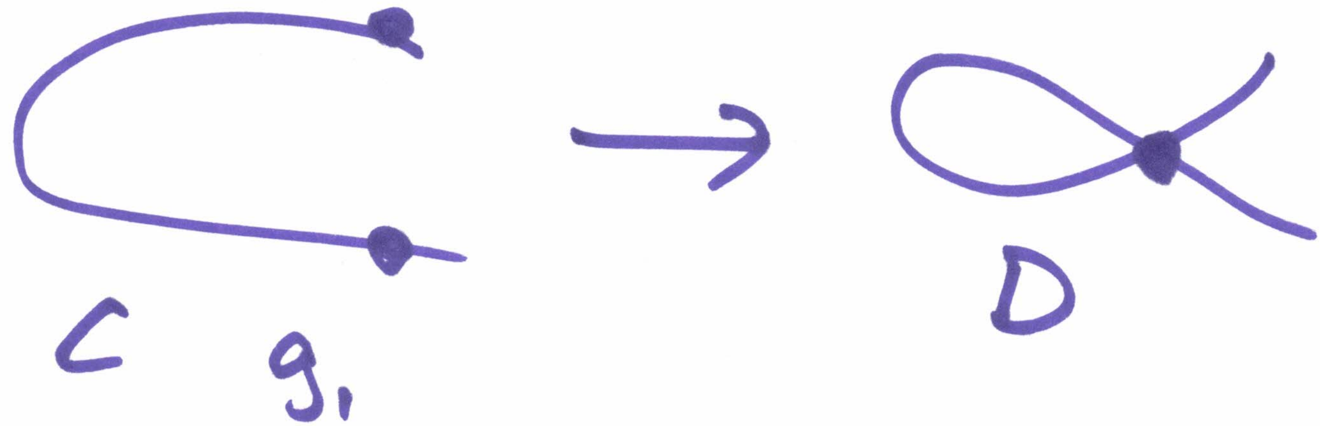
2.7







2.2 non-compact type



$\pi: \overline{M}_{g_1,2} \to \overline{M}_{g_1+1}$

Image called Δ_0 boundary divisor

$1 \to T \to \mathrm{Jac}(D) \to \mathrm{Jac}(C) \to 1$

alg torus — semi-abelian variety

$\delta M_g = \overline{M}_g - M_g = D_0 \cup \bigcup_{i=1}^{\lfloor g/2 \rfloor} D_i$

Question: Avoid the boundary. What is largest dimension of a complete subvariety of $\mathcal{M}_g$, $\mathcal{M}_g^{ct}$ or $\mathcal{A}_g$?

~~$\mathcal{M}$~~ moduli

If $Z \subset \mathcal{M}$ complete then $\dim(Z) \leq$

$U_{\mathcal{M}}$

$\mathcal{M}$	$U_{\mathcal{M}}$	
$\mathcal{M}_g$	$g-2$	Diaz
$g=4$ $\mathcal{M}_g^{ct}$	$2g-3$ 5	"
$\mathcal{A}_g$	$g(g-1)/2$	Van der Geer

$\Rightarrow$ by 1

↑ Keel Sadun: decrease $\mathbb{C}$

2.3 Ex $g=4$

$\mathcal{M}_4$ $\quad u_M = 2$ not known

$\mathcal{A}_4$ $\quad u_M = 5$ over $\mathbb{C}$ not known

$\qquad\qquad 6$ over $\overline{\mathbb{F}}_p$ known

$\mathcal{M}_4^{ct}$

$\quad \underline{u_M = 4}$ over $\mathbb{C}$ not known

$\qquad\qquad 5$ over $\overline{\mathbb{F}}_p$ known

upper bound for dim of complete family realized

2.5/ Over $\mathbb{C}$

Keel - Sadun: there is no complete subvariety of M_g^{ct} or A_g that has codim g

Over $\mathbb{F}_p$ such a subvariety exists.

2.5 p-rank

X p.p. abelian variety $/\overline{\mathbb{F}_p}$

$$\# X[p](k) = p^f \leftarrow \text{p-rank}$$

$$0 \leq f \leq g$$

$$f = \dim_{\mathbb{F}_p}(\mathrm{Hom}(\mu_p, X))$$

X semi-abelian var w/ toric part

then $f_X > 0$

If $f_X = 0$, no toric part

2.6. $\overline{\mathcal{M}}_g^0 \subset \mathcal{M}_g$

locus of
smooth curves
genus g
p-rank 0

Norman-Oort
$\operatorname{codim}(A_g^0, A_g) = g$

$A_g^0 \subset A_g$

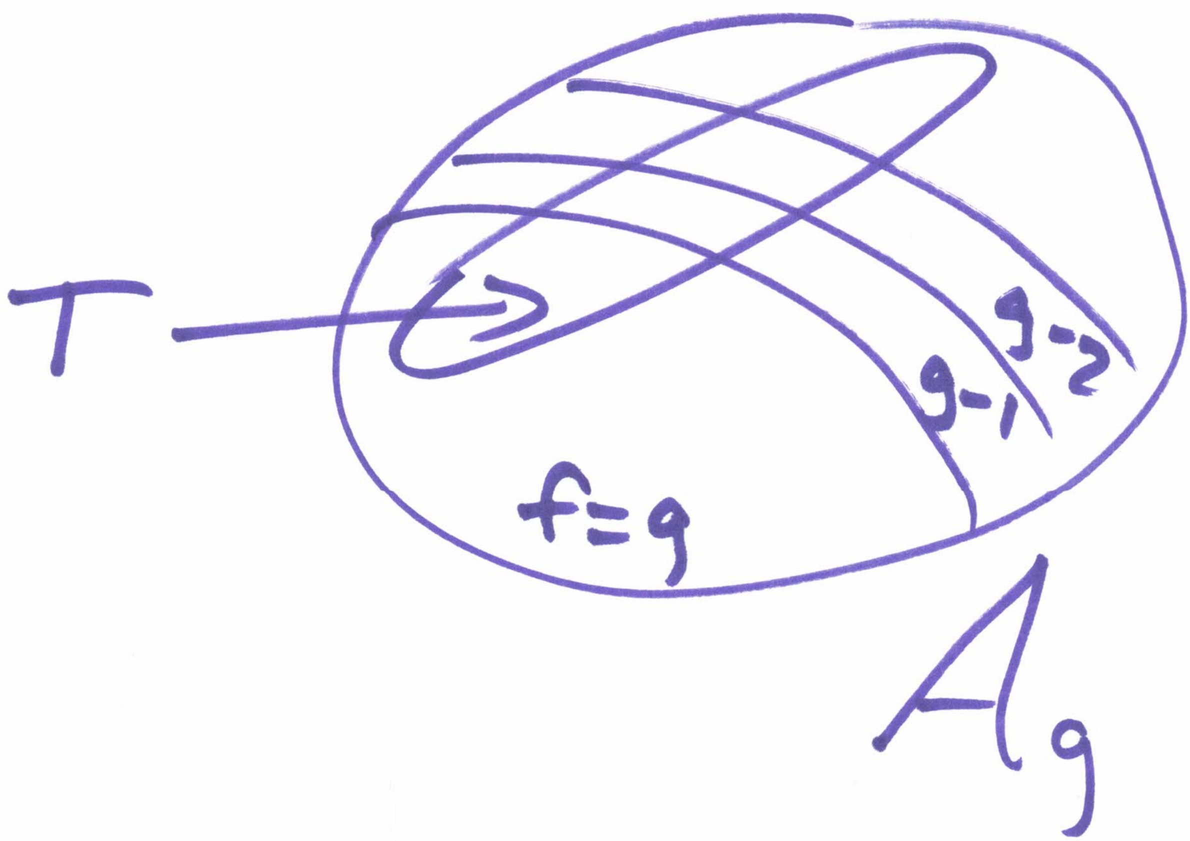
...

p-rank 0

Thm: Faber Van der Geer
$\operatorname{codim}(\overline{\mathcal{M}}_g^0, \mathcal{M}_g) = g$

2.6 idea:

typical point is ordinary
each time you decrease p-rank,
dimension goes down by 1



2.4 In M_g, there is a complete curve.

1-dim family of curves that does not hit boundary

Proof: $g=4$ $\quad r=2$

concrete $\quad$ RH: $\text{genus}(Z)=4$

$$\begin{array}{ll} Z & \\ \swarrow \ \text{2-to-1} & \text{branched at pair of 2 points in } W_C \\ C: y^2 = x^6 - 1 & W_C \subset C^2 - D_C \ \text{preimage} \\ \swarrow \ \text{2-to-1} & \\ E: y^2 = x^3 - 1 & O_E,\ Q \end{array}$$

$$W \subset E^2 - \Delta_E$$

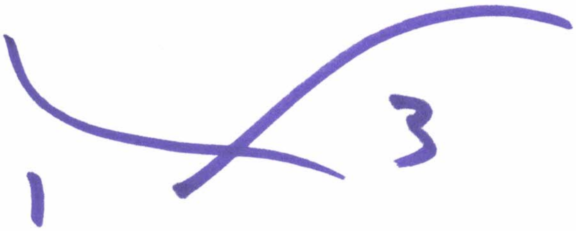
$$\{(P, P+Q)\}_{P \in E}$$

2.4 cont

Abstract

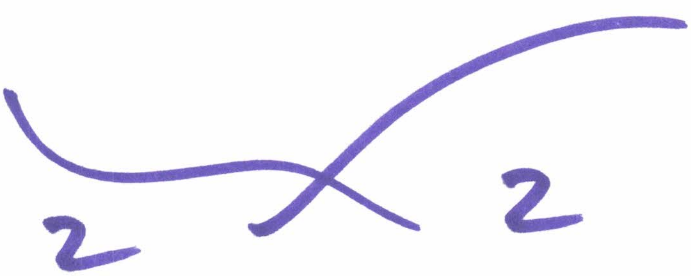
$$M_4^{ct} \longrightarrow A_4$$

minimal compactification



$X_1 \oplus X_3$

codim 2



$X_2 \oplus X_2$

codim 2

the error on this slide (and the transcript) is due to the lack of caffeination of the speaker.
The main point is the construction of the minimal compactification of A_4 & the fact that the image of the boundary has codim ≥ 2, so a typical dimension 1 family of M_4 does not intersect it.

2.7

Thm: Kudo/Harashita/Senda

$\forall p$, $\exists$ smooth curve C of genus $g=4$ that is supersingular

$a_C \geq 3$

$J_C \sim E^4$, E ss

new proof:

$$\dim 9 \longrightarrow T \subset \mathcal{A}_4 \quad \dim 10$$

Torelli

Jacobians of curves of compact type

$$T \subset \mathcal{A}_4 \quad \text{dim } 10$$
codim

$$\downarrow$$

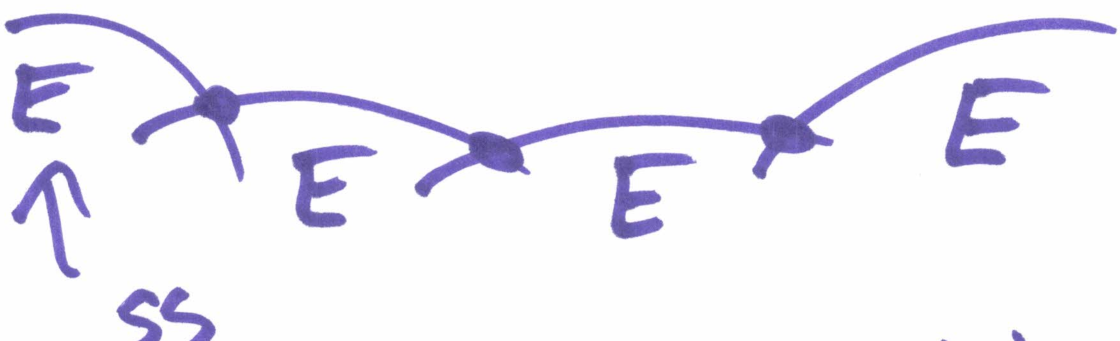
$$\dim(\Gamma) \geq 3 \qquad \mathcal{A}_4^{0} \quad \text{dim } 6$$

$$\downarrow$$

$$\Gamma = \boxed{T \cap \mathcal{A}_4[ss]} \qquad \mathcal{A}_4[ss]$$

Li/Oort
dim 4

<u>step 1</u> Γ non-empty



C
genus 4

$$\mathrm{Jac}(C) = E^4$$
supersing

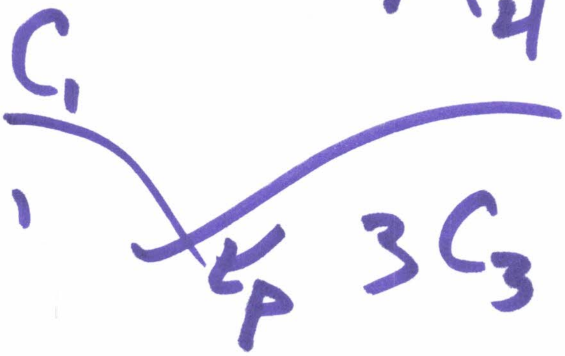
step 2

Jacobians of curves $\Gamma^{sing} \subset \Gamma$

singular & compact type

$$\dim(\Gamma^{sing}) \le 2$$

$$\overline{M}_4 \longrightarrow A_4$$



$\longrightarrow J(C_1) \oplus J(C_3)$

0-dim dim2

$\longrightarrow J(C_1) \oplus J(C_2)$

dim1 dim1

$$\dim(\Gamma^{sing}) = 2$$

Conclusion:

typical point in Γ

is Jac of C

smooth

$g=4$, supersingular

$\square$

Similar: $\forall\, p$

$\exists$ curve smooth genus 5

w/ slopes

$(1/4, 3/4) \oplus (1/2, 1/2)$