THE TORELLI LOCUS AND NEWTON POLYGONS
AWS 2024: COURSE AND PROJECT OUTLINE

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This lecture series is about the Torelli locus in the moduli space of abelian varieties, with applications to Newton polygons of curves in positive characteristic. In general, the lectures will cover two topics: the first is about the *geometry* of the Torelli locus; the second is about the *arithmetic* invariants of abelian varieties that occur for Jacobians of smooth curves in positive characteristic.

This is a preliminary document which will be expanded and refined later. In particular, a more complete bibliography will be provided at a later time.

1. The Torelli locus

Let g be a positive integer. Suppose X is a (smooth, projective, connected) curve of genus g. The Jacobian J_X of X is the quotient of the group of divisors of degree zero by the subgroup of principal divisors. One can show that the Jacobian J_X is a (principally polarized) abelian variety of dimension g. Many facts about X are determined by its Jacobian; for example, the unramified cyclic degree ℓ covers of X are determined by ℓ-torsion points on the Jacobian J_X.

For $1 \leq g \leq 3$, almost every abelian variety is a Jacobian. For example, an abelian variety of dimension $g = 1$ is an elliptic curve. An abelian surface (resp. threefold) is the Jacobian of a smooth curve of genus 2 (resp. 3) unless it decomposes as a product, together with the product polarization.

For $g \geq 4$, the situation is more interesting because not every abelian variety is a Jacobian. There are several methods to determine which abelian varieties are Jacobians but these are fairly difficult. It is often possible to study Jacobians of curves in a more explicit and concrete way than for a typical abelian variety. On the other hand, there are techniques for studying families of abelian varieties that do not apply when studying families of Jacobians of curves. This leads to a very valuable and rewarding exchange between these topics.

Consider the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. Within $\mathcal{A}_g$, we can consider the Torelli locus whose points represent Jacobians of curves. This sublocus of $\mathcal{A}_g$ has essential importance and plays an important role in many problems. Let $\mathcal{M}_g$ denote the moduli space of (smooth, projective, connected) curves of genus g. The Torelli morphism $\tau : \mathcal{M}_g \to \mathcal{A}_g$ takes a curve X to its Jacobian. It is an embedding, meaning that X is uniquely determined by J_X. The *open Torelli locus* $\mathcal{T}_g^\circ$ is the image of τ; it is the locus of all principally polarized abelian varieties of dimension g that are Jacobians.

When $g = 1, 2, 3$, then $\mathcal{T}_g^\circ$ is open and dense in $\mathcal{A}_g$, meaning that almost every principally polarized abelian variety of dimension $g \leq 3$ is a Jacobian. For $g \geq 2$, the dimension of $\mathcal{M}_g$ is $3g - 3$, while the dimension of $\mathcal{A}_g$ is $g(g + 1)/2$. So, as g increases, the open Torelli locus has increasingly high codimension in $\mathcal{A}_g$.

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2. The boundary

Surprisingly, some facts about smooth curves can be proven using singular curves; some facts about principally polarized abelian varieties that are indecomposable can be proven using principally polarized abelian varieties that decompose. For this reason, it is useful to consider compactifications of these moduli spaces, namely the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ and the toroidal compactification $\tilde{\mathcal{A}}_g$ of $\mathcal{A}_g$.

The points of the boundary of $\mathcal{M}_g$ represent stable singular curves, which are either of compact or non-compact type. When the dual graph of a curve is a tree, we say that the curve has compact type. To construct a singular curve of compact type, we take two curves, choose a point on each, and identify these in an ordinary double point. If $g_1 + g_2 = g$, this yields a morphism:

$$\kappa_{g_1,g_2} : \overline{\mathcal{M}}_{g_1;1} \times \overline{\mathcal{M}}_{g_2;1} \to \overline{\mathcal{M}}_g.$$

The Jacobian of a singular curve of compact type is an abelian variety, although it does decompose together with the product polarization.

To construct a singular curve of non-compact type, we take a curve, choose two points on it, and identify these in an ordinary double point. This yields a morphism:

$$\kappa_0 : \overline{\mathcal{M}}_{g-1;2} \to \overline{\mathcal{M}}_g.$$

The Jacobian of a singular curve of non-compact type is a semi-abelian variety. Later notes will include more description of semi-abelian varieties, including the toroidal rank of a semi-abelian variety and the toroidal compactification $\tilde{A}_g$. We extend the Torelli morphism $\tau : \overline{\mathcal{M}}_g \to \tilde{\mathcal{A}}_g$.

Historically, many statements about the geometry of $\mathcal{M}_g$ use the morphisms κ_{g_1,g_2}, κ_0, which are called clutching morphisms. The Torelli map extends to a map $\overline{\tau} : \overline{\mathcal{M}}_g \to \tilde{\mathcal{A}}_g$. However, $\overline{\tau}$ is no longer an embedding; in fact, some of its fibers have positive dimension.

3. Arithmetic invariants

Over a field of positive characteristic p, an elliptic curve can be ordinary or supersingular. We say that an elliptic curve is ordinary if it has point of order p; alternatively, an elliptic curve is ordinary if its Newton polygon has slopes of zero and one. Otherwise, the elliptic curve is supersingular. There are many results about ordinary and supersingular elliptic curves, due to Deuring [Deu41] and Igusa [Igu58]; for example, for a fixed prime p, most elliptic curves are ordinary and the number of isomorphism classes of supersingular elliptic curves is approximately $p/12$. See also [Man61].

In positive characteristic, the action of Frobenius determines important information about an abelian variety. To keep track of this information, there are combinatorial invariants called the p-rank, Newton polygon, the Ekedahl–Oort type, and the a-number. For an abelian variety A, the p-rank is the integer f such that the number of p-torsion points on A equals p^f. The Newton polygon is the Newton polygon of the L-polynomial, which is the characteristic polynomial of Frobenius on the crystalline cohomology; when $A = J_X$ for a curve X, the Newton polygon keeps track of the number of points on X defined over finite fields of characteristic p. The Ekedahl–Oort type is an invariant that classifies the structure of the p-torsion group scheme $A[p]$ of A; when $A = J_X$, this is the same as the structure of the de Rham cohomology as a module under Frobenius F and Verschiebung V. The a-number is the number of generators of $A[p]$ as a module under F and V.

The possibilities for the Newton polygon and Ekedahl–Oort type of an abelian variety are well understood. In contrast, in most cases it is not known which Newton polygons and Ekedahl–Oort types occur for Jacobians of curves for a given prime p. Some Newton polygons and Ekedahl–Oort types have been shown to occur for Jacobians and some Ekedahl–Oort types have been ruled out. More generally, the stratifications of $\mathcal{A}_g$ by these invariants are well understood; however, it is not understood how these stratifications intersect the Torelli locus. As applications of the theory covered in this lecture series, I will show how the geometric techniques used to study moduli spaces can shed light on these questions.

Lectures:
Here is a tentative schedule of lectures. These lectures are about abelian varieties defined over an algebraically closed field. The first half of each lecture includes material that makes sense for fields of any characteristic; the second half of each lecture includes applications for abelian varieties in positive characteristic.

(1) **The Torelli locus and arithmetic invariants**

In the first half of this lecture, I will give several descriptions of the Torelli locus in the moduli space $\mathcal{A}_g$ of abelian varieties of dimension g. With a dimension count, we can see that the Torelli locus is open and dense inside $\mathcal{A}_g$ when $1 \leq g \leq 3$, and has positive codimension for $g \geq 4$.

In the second half of this lecture, I will describe some arithmetic invariants of abelian varieties in positive characteristic p. These include: the p-rank, the Newton polygon, the Ekedahl–Oort type, and the a-number, see [Pri19] for a survey. As some applications, we can see the proofs of these facts, for every prime p:
(i) there exists an ordinary smooth curve of every genus g, [Mil72];
(ii) there exists a non-ordinary smooth curve of every genus g; and
(iii) there exists a supersingular curve of genus 2 [Ser83], [IKO86].

The proofs make use of the Cartier operator.

(2) **The boundary of the moduli spaces of curves and abelian varieties**

In the first half of this lecture, I will describe the boundary of the moduli space of curves and the clutching morphisms, as described in Section 2. The boundary is the image of the clutching morphisms, whose domain consists of products of moduli spaces of curves with marked points. Then we will cover some results of Diaz [Dia84] and Looijenga [Loo95] that show that a subspace $S \subset \overline{\mathcal{M}}_g$ having codimension at most g must intersect the boundary.

In the second half of this lecture, I will describe the purity result of de Jong and Oort [dJO00] for the Newton polygon stratification of $\mathcal{A}_g$. As an application, for every prime p, this yields a proof that there exists a supersingular curve of genus 3 [Oor91], and a supersingular curve of genus 4 [KHS20], [Pri]. We will see that this proof does not extend to curves of higher genus.

(3) **Special families of abelian varieties**

In the first half of this lecture, I will describe the situation for abelian varieties having additional structure; namely, whose automorphism group contains a cyclic group. The moduli spaces of these provide examples of Deligne–Mostow Shimura varieties. We say this moduli space is *special* if an open and dense subset of a component of the Shimura variety is contained in the Torelli locus. In particular, we consider families of Jacobians of curves that are cyclic covers of the projective line. The families that have special moduli spaces were classified by Moonen [Moo10]. The situation for Jacobians of abelian covers of the projective line is not fully understood and is related to a conjecture of Coleman and Oort.

In the second half of this lecture, I will describe constraints on the Newton polygon and Ekedahl–Oort type of an abelian variety in these special families. As an application, this shows that there exist supersingular curves of genus 5, 6, and 7, under certain congruence conditions on the prime p [LMPT19]. Furthermore, I will describe the rate of growth of the number of non-ordinary curves in these families [CP].

(4) **Inductive systems of moduli spaces of curves**

In the first part of this lecture, I will describe inductive systems of moduli spaces of curves. Via the clutching morphisms, it is possible to study moduli spaces of curves by induction on the genus. Similarly, there are inductive systems of moduli spaces of curves that are cyclic covers of the projective line. However, it is a delicate problem to preserve arithmetic properties of the Jacobians of the curves when deforming away from the boundary.

In the second half of this lecture, I will explain how this technique can be used to study the p-rank stratification of $\mathcal{M}_g$ [FvdG04]. If time permits, we will see how the p-torsion and the ℓ-torsion are independent of each other, in a way that can be made precise using ℓ-adic monodromy groups of the p-rank stratification [AP08].

Projects:

Any information on these problems will lead to progress on more general open questions. Currently, for accessibility, they are written for special cases in which the answer is unknown. The problems will be described in more detail later.

(1) (Computational) Determine geometric properties of strata of the moduli space $\mathcal{M}_4$ (such as the non-ordinary locus, p-rank 0 locus, and supersingular locus) by counting the number of isomorphism classes of curves with given invariants over a finite field.
(2) (Computational) For $5 \leq g \leq 10$, determine the Newton polygons and Ekedahl–Oort types whose strata have codimension at most g, have codimension at most $3g-3$, and/or do not occur on the boundary of $\overline{\mathcal{M}}_g$.
(3) Determine the intersection of the supersingular locus of $\mathcal{M}_3$ with the boundary of $\mathcal{M}_3$; similar question for the hyperelliptic locus $\mathcal{H}_3$. Generalize to $\mathcal{M}_4$.
(4) For one-dimensional special families of abelian (non-cyclic) covers $X \to \mathbb{P}^1$: find the Newton polygons and Ekedahl–Oort types that occur for curves in these families; for primes such that the generic curve in the family is ordinary, find the rate of growth of the number of non-ordinary curves in the family.
(5) Study the p-rank stratification of the moduli space of double covers of a fixed elliptic curve with $2n$ branch points.

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