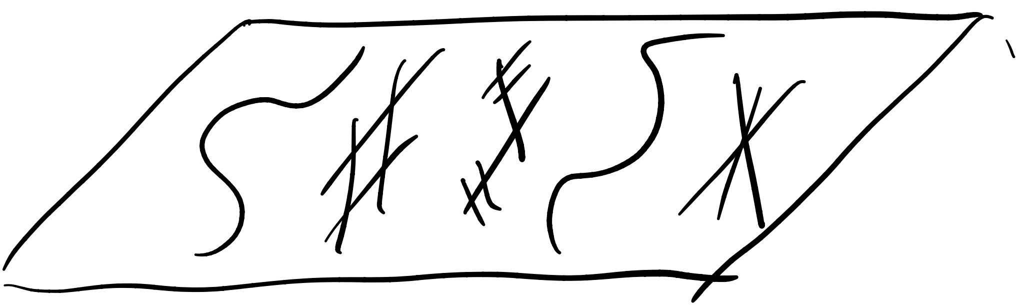
Silverman Lecture #2

I A Counting Application

II Local Heights



K number field

A/K abelian variety

$D \in \mathrm{Div}(A)$ symmetric ample

$$\hat{h}_D : A(K) \to [0, \infty)$$

$$\hat{h}_D(P) = \lim_{n \to \infty} \frac{1}{4^n} h_D(2^n P)$$

Thm $\hat{h}_D$ is a positive definite quad. form

$$\langle P, Q \rangle_D = \frac{1}{2} \left(\hat{h}_D(P+Q) - \hat{h}_D(P) - \hat{h}_D(Q) \right)$$

4

$$\langle P, P \rangle_D = \hat{h}_D(P)$$

pos. def. $A(K)/A(K)_{tors}$

$$A(K)_{\mathbb{R}} := A(K) \otimes \mathbb{R} \cong \mathbb{R}^{\operatorname{rank} A(K)}$$

$\hat{h}_D$ is pos. def q. form on $A(K)_{\mathbb{R}}$

$A(K)_{\mathbb{R}}$ element $\displaystyle\sum_{i=1}^{n} P_i \otimes c_i$, $\quad c_i \in \mathbb{R}$

$$\left\langle \sum_{i=1}^{n} P_i \otimes a_i , \sum_{j=1}^{m} Q_j \otimes b_j \right\rangle_D := \sum_{i,j} a_i b_j \langle P_i, Q_j \rangle_D$$

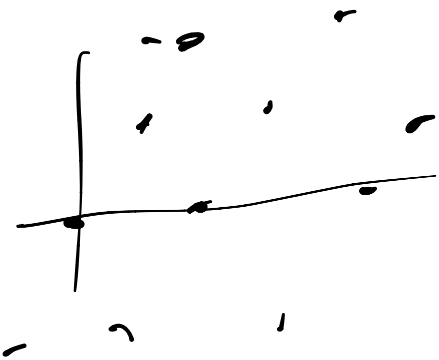
$$\sum c_i P_i$$

$\|\cdot\|_D : A(K)_{\mathbb{R}} \to [0,\infty)$, $\quad \|P\|_D = \sqrt{\langle P,P\rangle_D}$

$(A(K)_{\mathbb{R}}, \|\cdot\|_D)$ is a Euclidean $\mathbb{R}$-vector space

$$A(K)/A(K)_{tors} \hookrightarrow A(K)_{\mathbb{R}}$$

Image: $A(K)_{\mathbb{Z}}$ is a lattice in $A(K)_{\mathbb{R}}$



Theorem: (Néron)

A/K, D, h_D, $r = \operatorname{rank} A(K)$

$$\mathcal{N}(A(K), h_D, T)$$

$$:= \#\{P \in A(K) : h_D(P) \leq T\}$$

$$= \alpha(A/K, D)\, T^{r/2} + O(T^{\frac{r-1}{2}})$$

as $T \to \infty$

where

$$\underset{\shortparallel}{\alpha(A/K, D)} > 0$$

<u>Proof (sketch)</u>

(I) $\hat{h}_D = h_D + O(1)$

$\Rightarrow$ suffices to prove

$$N(A(K), \hat{h}_D, T) = \alpha T^{r/2} + O(T^{\frac{r-1}{2}})$$

$$\begin{tikzcd}
0 \arrow[r] & A(K)_{tor} \arrow[r] & A(K) \arrow[r] \arrow[dr] & A(K)_{\mathbb{R}} & \\
 & & & A(K)_{\mathbb{Z}} \arrow[u, phantom, sloped, "\subset"] \arrow[r] & 0
\end{tikzcd}$$

Exact

$$0 \to A(K)_{tors} \to A(K) \to A(K)_{\mathbb{Z}} \to 0$$

$$\begin{array}{c} A(K)_{\mathbb{Z}} \\ \cap\!| \\ A(K)_{\mathbb{R}} \end{array}$$

$$N(A(K), \hat{h}_D, T)$$

$$= \# A(K)_{tors} \cdot N(A(K)_{\mathbb{Z}}, \hat{h}_D, T)$$

because $\hat{h}_D(P) = 0 \iff P \in A(K)_{tors}$

$$= \# A(K)_{tors} \; N(A(K)_{\mathbb{Z}}, \|\cdot\|_D^2, T)$$

$$= \# A(K)_{tors} \; \underbrace{N(A(K)_{\mathbb{Z}}, \|\cdot\|_D, T^{1/2})}_{\text{count pts in a lattice in a Euclidean vector space}}$$

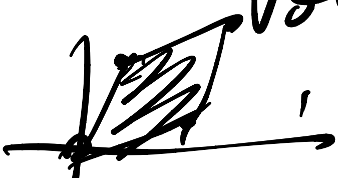
III Use a standard Thm

<u>Thm</u>: V $\mathbb{R}$-vector space
$\|\cdot\|$ Euclidean norm
$L \subset V$ lattice (full rank)

Then
$$\#\{v \in L : \|v\| \le T\} = \alpha T^{\dim V} + O(T^{\dim V - 1})$$

where

$$\alpha = \frac{\mathrm{vol}\,\{v \in V : \|v\| \le 1\}}{\mathrm{vol}\left(\begin{array}{c}\text{fundamental}\\ \text{domain for}\\ \text{the action of}\\ L \text{ on } V\end{array}\right)}$$



(IV) Apply Theorem

$V = A(K)_{\mathbb{R}}$

$L = A(K)_{\mathbb{Z}}$

$\| \cdot \| = \| \cdot \|_D$

$\Longrightarrow$ Néron's Theorem w Th

$$\alpha(A/K, D) = \frac{\#A(K)_{tors} \cdot \mathrm{Vol}\left(\begin{matrix}\text{unit ball in}\\ \mathbb{R}^n\end{matrix}\right)}{\mathrm{Reg}(A/K)^{1/2}}$$

$\mathrm{Reg}(A/K) = \det(\langle P_i, P_j \rangle_D)$

QED

Local Heights

v a place of K, $v \in M_K$

K_v = completion

$P \in \mathbb{P}^N(K_v)$

$D \in \mathrm{Div}(\mathbb{P}^N)$ effective

$$\boxed{\begin{array}{c} K=\mathbb{Q} \\ v=p \text{ or } \infty \\ K_v=\mathbb{Q}_p \text{ or } \mathbb{R} \end{array}}$$

Local ht:

$\lambda_{D,v}(P) :=$ v-adic local ht of P w.r.t. D

$= -\log\left(\begin{array}{l}\text{v-adic distance} \\ \text{from } P \text{ to } D\end{array}\right)$

Note:

$\lambda_{D,v}(P)$ is large $\Longleftrightarrow$ P is v-adically close to D

$\lambda_{D,v}(P) = \infty$

if $P \in |D|$

$$D = \{ F(\underline{x}) = 0 \}$$

↑ homog of deg d

in $K_v[x_0, \dots, x_N]$

$$\lambda_{D,v}(P) = -\log\min\left\{ \left|\frac{F(P)}{x_0(P)^d}\right|_v, \dots, \left|\frac{F(P)}{x_N(P)^d}\right|_v \right\}$$

Fact:

$$h_{\mathbb{P}^N, D}(P) = \sum_{v \in M_K} \lambda_{D,v}(P) + O(1)$$

$\forall P \in \mathbb{P}^N(K) \setminus |D|$

In general one can define local Weil hts

$$\lambda_{X,D,v} : X(K_v) \setminus |D| \to \mathbb{R}$$

$$h_D = \sum_v \lambda_{D,v} + O(1)$$

Néron constructed canonical local hts on abelian varieties.

Theorem (Néron)

$\exists$, (unique up to constants)

$$\hat{\lambda}_{D,v} : A(K_v) \setminus |D| \to \mathbb{R}$$

s.t.

(a) $\hat{\lambda}_{D,v}$ continuous for v-adic topology on $A(K_v)$

(b) $$\hat{\lambda}_{D+D',v} = \hat{\lambda}_{D,v} + \hat{\lambda}_{D',v} + \gamma_{D,D',v}$$

constant depends on v, 0 for almost all v

(c) $\varphi : A \to A'$

$$\hat{\lambda}_{A,\varphi^* D,v} = \hat{\lambda}_{A',D,v} \circ \varphi + \gamma_{A,A',\varphi,D,v}$$

(2) $\hat{h}_{A,D} = \sum_{v \in M_K} \hat{\lambda}_{A,D,v} - \kappa(A,D)$

Néron's construction

(1) limit formulas

(2) v non-arch

intersection theory

(3) v arch. $A(\mathbb{C}) = \mathbb{C}^g / L$

$\hat{\lambda} \sim \log |\text{theta function slightly modified}|$