Outline of Lectures and Projects for Canonical Heights on Abelian Varieties Arizona Winter School March 2–6, 2024

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Course Summary:
Height functions are used to measure the arithmetic (number theoretic) complexity of objects that are of interest to number theorists and arithmetic geometers. Very roughly,[1] you might think of the height of an object $\mathcal{O}$ as

$$h(\mathcal{O}) = \text{number of bits it takes to store } \mathcal{O} \text{ on a computer.}$$

Heights are used to prove finiteness results by showing that all of the objects in a set of objects have bounded height. They are also used to study the size of infinite sets of objects $\mathcal{S}$ by analyzing the growth rate of the height counting function

$$N(\mathcal{S}, T) = \#\bigl\{\mathcal{O} \in \mathcal{S} : h(\mathcal{O}) \le T\bigr\}.$$

We will be particularly interested in height functions on the set of points of an abelian variety. Thus let $K/\mathbb{Q}$ be a number field, and let A/K be an abelian variety. We start with height functions

$$h : A(K) \longrightarrow [0, \infty[$$

that measure the arithmetic complexity of the points of the Mordell–Weil group $A(K)$, and we will show that h interacts reasonably nicely with the group law on A. For example, the height h satisfies an approximate duplication formula

$$h(2P) = 4h(P) + (\text{quantity bounded independent of } P).$$

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[1]The "very roughly" means that we won't worry about scaling. So for example, people usually define the height of a rational number $a/b \in \mathbb{Q}$ written in lowest terms to be $h(a/b) = \log\max\bigl\{|a|, |b|\bigr\}$, even though it takes $\log_2|a| + \log_2|b| + 1$ bits to store a and b and a sign bit.

That's all well and good, but the "quantity bounded independent of P" is annoying! Following Néron and Tate, we will use h to construct a *canonical height*

$$\hat{h} : A(K) \longrightarrow [0, \infty[.$$

The canonical height $\hat{h}$ differs from h by a bounded amount, which means that $\hat{h}$ still measures arithmetic complexity, but $\hat{h}$ satisfies an exact duplication formula

$$\hat{h}(2P) = 4\hat{h}(P).$$

Even better, the canonical height $\hat{h}$ extends to give a positive definite quadratic form

$$\hat{h} : A(K) \otimes \mathbb{R} \cong \mathbb{R}^{\operatorname{rank} A(K)} \longrightarrow [0, \infty[,$$

and $A(K)/A(K)_{\mathrm{tors}}$ sits as a lattice (discrete subgroup) of the real vector space $\mathbb{R}^{\operatorname{rank} A(K)}$. This means thatw we now have at our disposal all of the wonderful tools from the theory of lattices in $\mathbb{R}^n$ to study the group of rational points $A(K)$.

Lecture 1: Construction and properties of canonical heights

We will construct Weil height functions on projective space $\mathbb{P}^N(K)$ and its subvarieties, and we will descibe some reasonably nice transformation properties that Weil heights have relative to morphisms $f : X \to Y$ of projective varieties. For example, on $\mathbb{P}^N$ we have the relation

$$h\bigl(f(P)\bigr) = \deg(f) \cdot h(P) + (\text{quantity bounded independent of } P).$$

We will use the Weil height on an abelain variety A to construct a canonical height $\hat{h}_A$ whose transformation formulas relative to isogenies $A \to A$ no longer require the "quantity bounded independent of P" error term.

Lecture 2: Applications; Local canonical heights

We will start with an application of the theory of canonical heights to the height counting function. Let A/K be an abelian variety defined over a number field, and let $r = \operatorname{rank} A(K)$ be the rank of its Mordell–Weil group. Then we will prove that[2]

$$\begin{aligned} N\bigl(A(K),T\bigr) &:= \#\bigl\{P \in A(K) : \hat{h}(P) \le T\bigr\} \\ &= \alpha(A/K) \cdot T^{r/2} + O(T^{(r-1)/2}) \quad \text{as } T \to \infty. \end{aligned}$$

We next turn to the local decomposition of the canonical height. Although the theory of local canonical heights is a bit technical, it is an invaluable tool. For each place v of K, let K_v denote the completion of K at v. Then the *local canonical height associated to* v is a v-adically continuous function[3]

$$\hat{\lambda}_v : A(K_v) \longrightarrow \mathbb{R}$$

that satisfies a quasi-duplication formula of the form

$$\hat{\lambda}_v(2P) = 4\hat{\lambda}_v(P) + \log\bigl|F(P)\bigr|_v + c_v,$$

where the function $F \in K(A)$ does not depend on v and the constant c_v does not depend on P. This allows us use analytic tools to study the individual local canonical heights $\hat{\lambda}_v$, and then use the formula

$$\hat{h}(P) = \sum_v \hat{\lambda}_v(P) + (\text{constant independent of } P)$$

to deduce information about the global canonical height $\hat{h}$.

Lecture 3: Lower bounds for canonical heights

Let A/K be an abelian variety defined over a number field. The canonical height $\hat{h} : A(K) \to [0, \infty[$ has the following agreeable property:

$$\hat{h}(P) = 0 \quad \Longleftrightarrow \quad P \text{ is a torsion point.}$$

This raises the question of how small $\hat{h}(P)$ can be if P is not a torsion point, or in more evocative language:

What is the smallest possible arithmetic complexity of a non-torsion point on an abelian variety?

[2]Proving this formula was one of Néron's motivations in constructing canonical heights.

[3]We are cheating here, and in a fairly serious way, since $\hat{\lambda}_v$ is really only defined on a Zariski open subset $U \subset A$, and $\hat{\lambda}_v$ has a logarithmic singularity as P approaches the v-adic boundary of $U(K_v)$ in $A(K_v)$. However, we've also undersold $\hat{\lambda}_v$ in the case that v is archimedean, since in that case $\hat{\lambda}_v$ is not merely continuous, it is a real-analytic function on a Zariski open subset of $A(\mathbb{C})$.

This question actually comes in two flavors: we can fix the abelian variety and vary the field, or we can fix the field and vary the abelian variety. These two directions lead to (generalizations) of two famous conjectures:

- [**Lehmer Conjecture**] Fix the abelian variety A/K. How small can $\hat{h}(P)$ be as a function of the degree $\bigl[K(P):K\bigr]$ of the field generated by the coordinates of P, as P ranges over all non-torsion points $P \in A(\bar{K})$. One might hope for an answer of the form
$$\hat{h}(P) \ge \frac{C(A/K)}{\bigl[K(P):K\bigr]^{\epsilon}}$$
for some small value of ϵ. The best possible value would be $\epsilon = 1/\dim(A)$. The best known value is a small multiple of $\dim(A)$.
- [**Lang Conjecture**] Fix the field K. How small can $\hat{h}(P)$ be as a function of the complexity of A, as A/K ranges over all abelian varieties defined over K and P ranges over all non-torsion points of $A(K)$. One might hope for an answer of the form
$$\hat{h}(P) \ge c_1(K)h(A/K) - c_2(K),$$
where $h(A/K)$ measures the arithmetic complexity of the abelian variety A.

In this lecture I'll describe these conjectures, as well as the classical Lehmer conjecture for $\bar{\mathbb{Q}}^*$, discuss some of the known results, and as time permits, sketch one or more of the proof methods used to tackle these sorts of height lower bound problems.

Lecture 4: Canonical heights in families; Specialization theorems

In this lecture we look at families of points P_t on families of abelian varieties A_t and study how the canonical heights $\hat{h}(P_t)$ vary as a function of the parameter t. For concreteness, here's an example of a family of points on a family of elliptic curves:
$$E_t : y^2 = x^3 - (t^2 - t)x + 2t + 1, \quad P_t = (t, t+1).$$
More generally, we look at a family of abelian varieties parameterized by a curve,
$$A \longrightarrow C, \quad \text{and a family of points} \quad P : C \longrightarrow A.$$
The family of points P may be viewed as a single point $P \in A\bigl(K(C)\bigr)$ defined over the function field of C, so P has a an associated function field canonical height $\hat{h}(P)$. Then, for each point $t \in C(\bar{K})$, we

get a point P_t on the abelian variety $A_t(\bar{K})$ that has its canonical height $\hat{h}(P_t)$. Finally, we can fix a height function on the points $C(\bar{K})$ of the curve C. Our goal in this lecture is to sketch a proof of the limit formula

$$\lim_{\substack{t\in C(\bar{K})\\ h(t)\to\infty}} \frac{\hat{h}(P_t)}{h(t)} = \hat{h}(P).$$

We will then explain how the limit formula implies that the specialization map

$$A(C) \longrightarrow A_t(\bar{K}), \quad P \longrightarrow P_t,$$

is injective except for a set of points $t \in C(\bar{K})$ of bounded height. As time permits, we will discuss the general philosophy of "unlikely intersections," which predicts that if $\dim(A) \geq 2$, then the set of exceptional $t \in C(\bar{K})$ is not merely of bounded height, but is actually finite.

Project 1: Trace relations on abelian varieties

Let A/K be an abelian variety defined over a field K, and let L/K be finite a Galois extension with Galois group $G(L/K)$. Then we can define a trace map from $A(L)$ to $A(K)$ by the adding up the Galois conjugates of a point,

$$\mathrm{Trace}_{A,L/K} : A(L) \longrightarrow A(K), \quad \mathrm{Trace}_{A,L/K}(P) = \sum_{\sigma\in G(L/K)} \sigma(P).$$

In this project we will investigate this trace map. There are many natual questions, for example:

(a) Find a criterion for when $\mathrm{Trace}_{A,L/K}$ is surjective. We may consider this problem when K is a number field, a function field, a local field such as $\mathbb{Q}_p$, or a finite field.

(b) Let K be a number field or function field, and suppose that

$$\mathrm{Trace}_{A,L_v/K_v} : A(L_v) \longrightarrow A(K_v)$$

is surjective for every completion of L/K. Does this imply that $\mathrm{Trace}_{A,L/K}$ is surjective? If not, what are some obstructions, and is their vanishing sufficient to ensure surjectivity.

(c) Work on questions (a) and (b) for quadratic extensions, i.e., extensions L/K with $[L : K] = 2$. We note that this case may be easier to handle, since there is a "twist abelian variety" A^χ/K so that

$$A(L) = A(K) \oplus A^\chi(K).$$

(d) What, if anything, can we say whe we combine the trace map and the canonical height function,

$$A(L) \longrightarrow \mathbb{R}, \quad P \longmapsto \hat{h}_{A,D}\bigl(\operatorname{Trace}_{A,L/K}(P)\bigr)?$$

Or we might consider the map

$$A(L) \longrightarrow \mathbb{R}, \quad P \longmapsto \det\Bigl(\bigl\langle \sigma(P), \tau(P)\bigr\rangle_{A,D}\Bigr)_{\sigma,\tau\in G(L/K)},$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{A,D}$ is the canonical height pairing. This map is quite interesting, since if D is ample and symmetric, then it sends P to 0 if and only if the set of Galois conjugates $\bigl\{\sigma(P) : \sigma \in G(L/K)\bigr\}$ is $\mathbb{Z}$-linearly dependent.

Project 2: Northcot and Bogomolov fields for abelian varieties

Let $K/\mathbb{Q}$ be a number field, let A/K be an abelian variety, and let $\hat{h}_{A,D}$ be the canonical height on A relative to an ample symmetric divisor. In this project we investigate the group $A(L)$ over extensions L/K of infinite degree.

Definition: The field L has the *Northcott Property for A/K* if

$$\bigl\{P \in A(L) : \hat{h}_{A,D}(P) \le B\bigr\} \quad \text{is finite for all } B \ge 0.$$

Definition: The field L has the *Bogomolov Property for A/K* if there exists a constant $c(A,D,L) > 0$ such that every $P \in A(L)$ satisfies either

$$\hat{h}_{A,D}(P) = 0 \quad \text{or} \quad \hat{h}_{A,D}(P) \ge c(A,D,L).$$

One can show that the Northcott property implies the Bogomolov property. In this project we will look for abelian varieties and fields that satisfy these properties. For example, the Bogomolov property is known if we take $L = K^{\mathrm{ab}}$ to be the maximal abelian extension of K, i.e., if $K(P)/K$ is a Galois extension with abelian Galois group. Can we prove the Bogomolov property if $K(P)/K$ is a solvable extension? Or we might look at weakened versions of the two properties, for example allowing the lower bound in the Bogomolov property to depend on $\bigl[K(P) : K\bigr]$ in some explicit way. This leads to abelian variety versions of the Lehmer conjecture (see Lecture 3 and Project 3), which we might investigate for various types of abelian varieties A/K and various types of extensions L/K.

Project 3: Experimental investigations of Lehmer's conjecture for elliptic curves

Let $K/\mathbb{Q}$ be a number field, let A/K be an abelian variety, and let $\hat{h}_{A,D}$ be the canonical height on A relative to an ample symmetric divisor. Then for points $P \in A(\bar{K})$, we have

$$\hat{h}_{A,D}(P) = 0 \quad \iff \quad P \in A_{\text{tors}}.$$

The *Lehmer conjecture for abelian varieties* gives an estimate for how small $\hat{h}_{A,D}(P)$ can be if P is not a torsion point. The strongest version says that there is a constant $c(A,D) > 0$ so that

$$\hat{h}_{A,D}(P) \ge \frac{c(A,D)}{\bigl[K(P):K\bigr]^{1/\dim(A)}} \quad \text{for all } P \in A(\bar{K}) \smallsetminus A_{\text{tors}}.$$

The classical Lehmer conjecture is a similar statement for the multiplicative group. It asserts that there is an absolute constant $c > 0$ so that

$$h(\alpha) \ge \frac{c}{\bigl[\mathbb{Q}(\alpha):\mathbb{Q}\bigr]} \quad \text{for all } \alpha \in \bar{\mathbb{Q}}^* \text{ that are not roots of unity.}$$

There is a lot of data available for the classical Lehmer conjecture, including a conjectural value for the best possible c. But as far as I am aware, there are no comparable computations for any elliptic curve, much less a higher dimensional abelian variety. In this project we'll choose a convenient elliptic curve $E/\mathbb{Q}$ and search for non-torsion points $P \in E(\bar{\mathbb{Q}})$ of small degree such that $d(P)\hat{h}_E(P)$ is small. This will include using the local decomposition of the canonical height,

$$\hat{h}_E(P) = \sum \hat{\lambda}_{E,v}(P),$$

and explicit formulas/series for the local heights $\hat{\lambda}_{E,v}$. (To some extent, we may be able to use pre-programmed versions of $\hat{\lambda}_{E,v}$, especially for $E(\mathbb{R})$ and $E(\mathbb{C})$, in Sage, Magma, PARI-GP, etc.)

As a side project, it may be useful to develop an explicit estimate of the following form: For $E/\mathbb{Q}$ and non-torsion $P \in E(\bar{\mathbb{Q}})$, let $K = \mathbb{Q}(P)$ and $d = [K:\mathbb{Q}]$. Find explict constants $C_1(d) > 0$ and $C_2(d,E) \ge 0$ so that

$$\hat{h}_E(P) \ge C_1(d) \cdot \log\bigl|\mathrm{Disc}(K/\mathbb{Q})\bigr| - C_2(d,E).$$

We'll want to make $C_1(d)$ as large as possible and $C_2(d,E)$ as small as possible.

Background Material and References on Abelian Varieties

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