ARIZONA WINTER SCHOOL LECTURE NOTES ON p-ADIC AND MOTIVIC INTEGRATION

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1. p-adic integration

1.1. **The p-adic measure.** Let p be a prime number. We consider a field K with a valuation $\mathrm{ord} : K^\times \to \mathbf{Z}$, extended to K by $\mathrm{ord}(0) = \infty$. We denote by $\mathcal{O}_K$ the valuation ring $\mathcal{O}_K = \{x \in K | \mathrm{ord}(x) \geq 0\}$ and we fix an uniformizing parameter ϖ, that is, an element of valuation 1 in $\mathcal{O}_K$. The ring $\mathcal{O}_K$ is a local ring with maximal ideal $\mathcal{M}_K$ of $\mathcal{O}_K$ generated by ϖ. We shall assume the residue field $k := \mathcal{O}_K/\mathcal{M}_K$ is finite with $q = p^e$ elements. We endow K with a norm by setting $|x| := q^{-\mathrm{ord}(x)}$ for x in K. We shall furthermore assume K is complete for $|\_|$.
It follows in particular that the abelian groups $(K^n, +)$ are locally compact, hence they have a canonical Haar measure μ_n, unique up to multiplication by a non zero constant, so we may assume $\mu_n(\mathcal{O}_K^n) = 1$. The measure μ_n is the unique $\mathbf{R}$-valued Borel measure on K^n which is invariant by translation and such that $\mu_n(\mathcal{O}_K^n) = 1$. For instance the measure of $a + \varpi^m \mathcal{O}_K^n$ is q^{-mn}. For any measurable subset A of K^n and any λ in K, $\mu_n(\lambda A) = |\lambda|^n \mu_n(A)$. More generally, for every g in $\mathrm{GL}_n(K)$,

$$\mu_n(gA) = |\mathrm{det} g|\mu_n(A). \tag{1.1.1}$$

If f is, say, a K-analytic function on A, we set

$$\int_A |f|\mu_n := \int_A |f||dx| := \sum_{m\in\mathbf{Z}} \mu_n(\mathrm{ord}(f) = m)q^{-m},$$

assuming the series $\sum_{m\in\mathbf{Z}} \mu_n(\mathrm{ord}(f) = m)q^{-m}$ is convergent in $\mathbf{R}$. More generally, we define similarly $\int_A |f|^s|dx|$ by $\sum_{m\in\mathbf{Z}} \mu_n(\mathrm{ord}(f) = m)q^{-ms}$ whenever it makes sense. For instance, when $n = 1$, we have, for $s > 0$ in $\mathbf{R}$,

$$\begin{aligned}\int_{x\in\mathcal{O}_K,\mathrm{ord}(x)\geq m} |x|^s &= \sum_{j\geq m} q^{-sj} \int_{\mathrm{ord}(x)=j} |dx| = \sum_{j\geq m} q^{-sj}(q^{-j} - q^{-j-1}) \\ &= (1 - q^{-1})q^{-(s+1)m}/(1 - q^{-(s+1)}).\end{aligned} \tag{1.1.2}$$

1.2. **Integration on analytic varieties.** Formula (1.1.1) is a very special form of the following fundamental change of variables formula (see [34] p. 111):

1.2.1. **Proposition** (The p-adic change of variables formula). *Let U be an open subset of K^n and consider K-analytic functions $f_1, \ldots, f_n$ on U. Assume $f =$*

$(f_1, \ldots, f_n) : U \to K^n$ is a K-analytic isomorphism between U and an open subset V of K^n. Then, for every integrable function φ on V,

$$\int_V \varphi \mu_{n|V} = \int_U (\varphi \circ f) |\partial(f_1, \ldots, f_n)/\partial(x_1, \ldots, x_n)| \mu_{n|U},$$

where $\partial(f_1, \ldots, f_n)/\partial(x_1, \ldots, x_n)$ is the determinant of the jacobian matrix of f.

Let X be an n-dimensional smooth K-analytic manifold. One assigns to any K-analytic n-differential form ω on X a measure $\mu_\omega := |\omega|$ as follows. Take an atlas $\{(U, \phi_U)\}$ of X. Write $(\phi_U^{-1})^* \omega_{|U} = f_U dx_1 \wedge \cdots \wedge dx_n$. If A is small enough to be contained in some U, we set $\mu_\omega(A) := \int_{\phi_U(A)} |f_U||dx|$. It follows from the change of variables formula that the measure may be extended uniquely by additivity to any A in a way which is independent of the choice of the atlas.

1.3. **Rationality of a Poincaré series.** Let f be a polynomial in $\mathcal{O}_K[x_1, \ldots, x_n]$. Denote by N_m the number of elements x in $(\mathcal{O}_K/\varpi^{m+1}\mathcal{O}_K)^n$ such that $f(x) \equiv 0 \mod \varpi^{m+1}$ and set

$$Q(T) := \sum_{m \geq 0} N_m T^m.$$

When $K = \mathbf{Q}_p$, Borevich and Shafarevich conjectured that $Q(T)$ is always a rational function of T.

1.3.1**. Theorem** (Igusa)**.** *Assume K is of characteristic zero (i.e. K is a finite extension of $\mathbf{Q}_p$). Then the series $Q(T)$ is rational. More precisely it is of the form*

$$\frac{R(T)}{\prod_{j \in F}(1 - q^{-a_j} T^{b_j})}$$

with $R(T)$ in $\mathbf{Z}[p^{-1}][T]$, F finite, a_j in $\mathbf{N}$ and b_j in $\mathbf{N} \setminus \{0\}$.

The idea of Igusa's proof is the following. We refer to [33] or [34] for more details. One first observe that

$$N_m = q^{(m+1)n} \mu_n(\{x \in \mathcal{O}_K^n | \mathrm{ord} f(x) \geq m+1\}).$$

By an easy calculation similar to (1.1.2) one deduces the relation

$$Q(q^{-n-s}) = \frac{q^n}{1 - q^{-s}}(1 - I(s)),$$

with

$$I(s) := \int_{\mathcal{O}_K^n} |f|^s |dx|.$$

Hence it is sufficient to prove the rationality of $I(s)$ as function of q^{-s}. This is achieved in the following way. By Hironaka's resolution (this is the place where the hypothesis that K is of characteristic zero is crucial), there exists a smooth compact manifold Y and an analytic morphism $h : Y \to \mathcal{O}_K^n$, obtained by composition of blowing up smooth centers, which is an isomorphism away from the locus of $f = 0$. By the change of variables formula $I(s)$ may be expressed as

$$I(s) = \int_Y |f \circ h|^s |h^* dx|.$$

On Y, $f \circ h$ and $h^* dx$ are both locally monomial, i.e. of the form $f \circ h = u \prod y_i^{N_i}$ and $h^* dx = v \prod y_i^{n_i} dy$, with u and v units, and $y = (y_1, \dots, y_n)$ local coordinates, in which case the explicit calculation of the integral becomes very easy, since it is a product of integrals of type (1.1.2).

1.4. **The Serre series.** Instead of considering the number N_m of approximate solutions modulo ϖ^{m+1} of $f = 0$ in $(\mathcal{O}_K/\varpi^{m+1}\mathcal{O}_K)^n$, one may want to consider approximate solutions that can be lifted to actual solutions of $f = 0$ in $\mathcal{O}_K^n$. More precisely, we denote by $\tilde{N}_m$ the number of elements y in $(\mathcal{O}_K/\varpi^{m+1}\mathcal{O}_K)^n$ such that $y \equiv x \mod \varpi^{m+1}$, with x in $\mathcal{O}_K^n$ such that $f(x) = 0$. The corresponding generating series

$$P(T) := \sum_{m \geq 0} \tilde{N}_m T^m$$

was first considered by Serre who raised the question of its rationality, that was solved (in characteristic zero) by Denef in [9].

1.4.1. **Theorem** (Denef). *Assume K is of characteristic zero. Then the series $P(T)$ is rational. More precisely it is of the form*

$$\frac{R(T)}{\prod_{j \in F}(1 - q^{-a_j} T^{b_j})}$$

with $R(T)$ in $\mathbf{Z}[p^{-1}][T]$, F finite, a_j in $\mathbf{Z}$ and b_j in $\mathbf{N} \setminus \{0\}$.

Since

$$\tilde{N}_m = q^{(m+1)n} \mu_n(\{y \in \mathcal{O}_K^n | \exists x, f(x) = 0, x \equiv y \mod \varpi^{m+1}\}),$$

one can reduce the rationality of $P(T)$ to the rationality of the integral

$$J(s) := \int_{\mathcal{O}_K^n} d(x, V)^s |dx|,$$

where $d(x, V)$ is the distance function to the hypersurface V defined by $f = 0$. One then sees that a major difference with the Igusa case 1.3 occurs: the function $d(x, V)$ is in general not analytic, due to the presence of quantifiers in its definition. So, in the proof of his Theorem, Denef had to use Macintyre's Theorem on quantifier elimination, which we shall explain now.

Let us mention that, in the positive characteristic case, the rationality of $Q(T)$ would follow at once, as soon as Hironaka resolution will be known. For the rationality of $P(T)$, the situation is much more open, since, in this setting, one does not know, even conjecturally, what could be a sensible analogue of Macintyre's Theorem.

1.5. **Definable subsets of $\mathbf{Q}_p$.** For simplicity we shall assume $K = \mathbf{Q}_p$, the case of finite extensions of $\mathbf{Q}_p$ being quite similar.
Let $\mathcal{L}_{\mathrm{Mac}}$ denote the first order language whose variables run over $\mathbf{Q}_p$ and with symbols to denote $+, -\times, 0, 1$ and, for every $d = 2, 3, 4 \dots$, a symbol P_d to denote the predicate "x is a d-th power in $\mathbf{Q}_p$". Moreover, for every element in $\mathbf{Z}_p$, there is a symbol to denote that element. As for any first order language, formulas of $\mathcal{L}_{\mathrm{Mac}}$ are built up from the above specified symbols and variables, together with the logical connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not), the quantifiers $\exists$, $\forall$ and the equality symbol $=$. Macintyre's Theorem [36] states that $\mathbf{Q}_p$ has quantifier elimination in the language $\mathcal{L}_{\mathrm{Mac}}$, meaning that every formula in that language is equivalent in $\mathbf{Q}_p$ to a formula without quantifiers. A subset of $\mathbf{Q}_p^n$ is called semi-algebraic if is definable by a (quantifier-free) formula in $\mathcal{L}_{\mathrm{Mac}}$.
We shall also consider the first order language $\mathcal{L}_{\mathrm{Pres}}$ of Presburger arithmetic. In this language variables run over $\mathbf{Z}$ and symbols are $+, \leq, 0, 1$ and, for every $d = 2, 3, 4 \dots$, a symbol to denote the binary relation $x \equiv y \mod d$. One should note there is no symbol in $\mathcal{L}_{\mathrm{Pres}}$ for multiplication. It is an old result of Presburger that $\mathbf{Z}$ has quantifier elimination in the language $\mathcal{L}_{\mathrm{Pres}}$.
It is also useful to consider the first order language $\mathcal{L}$ with two sorts of variables: a first sort of variable running over $\mathbf{Q}_p$ and a second sort running over $\mathbf{Z}$. The symbols of $\mathcal{L}$ consist of the symbols of $\mathcal{L}_{\mathrm{Mac}}$ for the first sort, the symbols of $\mathcal{L}_{\mathrm{Pres}}$ for the second sort, and a symbol to denote the valuation function $\mathrm{ord} : \mathbf{Q}_p \setminus \{0\} \to \mathbf{Z}$. As remarked in [9], it follows from Macintyre's Theorem that $\mathbf{Q}_p$ has elimination of quantifiers in the language $\mathcal{L}$ and every subset of $\mathbf{Q}_p^n$ which is definable in $\mathcal{L}$ is semi-algebraic. A function is called $\mathcal{L}$-definable if its graph is $\mathcal{L}$-definable.

1.6. **Denef's Cell Decomposition Theorem.** In [9] Denef gave two proofs of Theorem 1.4.1. We already mentioned the first one, which uses Hironaka resolution and Macintyre's Theorem. The second one was based on the following cell decomposition Theorem 1.6.1 which Denef originally deduced from Macintyre's Theorem (in fact, Macintyre's Theorem also easily follows from Theorem 1.6.1). Then, in [11] Denef gave a direct proof of Theorem 1.6.1

1.6.1. **Theorem** (Denef's p-adic cell decomposition). *Let $f_i(x,t)$, $1 \leq i \leq m$, be polynomials in $\mathbf{Q}_p[x,t]$, with $x = (x_1, \dots, x_{n-1})$ and t another variable. Fix an integer $d \geq 2$. There exists a finite partition of $\mathbf{Q}_p^n$ into subsets (called cells) of the form*

$$A = \left\{ (x,t) \in \mathbf{Q}_p^n \;\middle|\; x \in C \text{ and } |a_1(x)| \square_1 |t - c(x)| \square_2 |a_2(x)| \right\},$$

where C is an $\mathcal{L}$-definable subset of $\mathbf{Q}_p^{n-1}$, $\square_i$ denotes either $\leq$, $<$, or no condition, and $a_i(x)$ and $c(x)$ are $\mathcal{L}$-definable functions from $\mathbf{Q}_p^{n-1}$ to $\mathbf{Q}_p$ such that, for every (x,t) in A,

$$f_i(x,t) = u_i(x,t)^d h_i(x) (t - c(x))^{\nu_i},$$

for $1 \leq i \leq m$, where $u_i(x,t)$ is a unit on A, $h_i(x)$ is an $\mathcal{L}$-definable function, and ν_i is in $\mathbf{N}$.

1.7. **Basic Theorem on p-adic integration and applications.** The following general result is proved by successive application of the cell decomposition Theorem and integration with respect to the t-variable, cf. [10] and [13].

1.7.1**. Theorem** (Denef)**.** *Let $(A_{\lambda,\ell})_{\lambda\in\mathbf{Q}_p^k,\ell\in\mathbf{Z}^r}$ be an $\mathcal{L}$-definable family of bounded subsets of $\mathbf{Q}_p^n$, meaning that the relation $x\in A_{\lambda,\ell}$ may be expressed by a formula in the language $\mathcal{L}$, with variables x,λ and ℓ. Let $\alpha(x,\lambda,\ell)$ be a $\mathbf{Z}$-valued $\mathcal{L}$-definable function on $\mathbf{Q}_p^n\times\mathbf{Q}_p^k\times\mathbf{Z}^r$. Assume that all values of α are ≥ 0. Then the integral*

$$I_{\lambda,\ell} := \int_{A_{\lambda,\ell}} p^{-\alpha(x,\lambda,\ell)}|dx|$$

is a $\mathbf{Q}$-valued function of λ,ℓ belonging to the $\mathbf{Q}$-algebra generated by functions of the form $\theta(\lambda,\ell)$ and $p^{\theta(\lambda,\ell)}$ with $\theta(\lambda,\ell)$ a $\mathbf{Z}$-valued $\mathcal{L}$-definable function.

In the special case where there is no variable λ, the function $I(\lambda)$ in Theorem 1.7.1 is particularly simple: it is built from Presburger functions (i.e. $\mathcal{L}_{\mathrm{Pres}}$-definable functions) using multiplication, exponentiation and $\mathbf{Q}$-linear combinations. From this observation Denef could deduce in an elementary way the following general rationality statement.

1.7.2**. Theorem** (Denef)**.** *Assume the notation of Theorem 1.7.1 with no λ involved. Then the series*

$$\sum_{\ell\in\mathbf{N}^r} I(\ell)T^\ell$$

in $\mathbf{Q}[[T_1,\ldots,T_r]]$ is a rational function of T.

Remark that the rationality of $Q(T)$ and $P(T)$ is a direct consequence of Theorem 1.7.2.
We now give a striking application to the problem of counting subgroups. For a finitely generated group G and an integer $n\geq 1$, let us denote by $a_n(G)$ the number number of subgroups of order n in G. This is always a finite number (cf. [30]).
The following Theorem is due to Grunewald, Segal and Smith [30]:

1.7.3**. Theorem.** *If G is a torsion-free finitely generated nilpotent group, then the series $\sum_m a_{p^m}(G)T^m$ is rational, for every prime p.*

The Theorem is proved by expressing $a_{p^m}(G)$ in terms of a p-adic integral

$$\int_{A_m} p^{-\theta(x)}|dx|,$$

with $(A_m)_{m\in\mathbf{N}}$ and θ definable in $\mathcal{L}$ and applying Theorem 1.7.2.

2. Prehistory of motivic integration: Proving results over $\mathbf{C}$ by computing p-adic integrals

2.1. **The topological zeta function.** Let X be a smooth complex algebraic variety of dimension n and D an effective divisor on X. By a log-resolution of (X,D) we mean a proper birational morphism $h: Y\to X$ with Y smooth, which is an

isomorphism away from the preimage of the support of D and such that $h^{-1}(D)$ a divisor with normal crossings. We denote by E_i, $i \in J$ the irreducible components of $h^{-1}(D)$ (in particular each E_i is smooth). We set $E_I = \cap_{i\in I} E_i$, for $I \subset J$ and $E_I^\circ = E_I \setminus \cup_{j\notin I} E_j$. Finally we denote by N_i the multiplicity of E_i in $h^{-1}(D)$, i.e. $h^{-1}(D) = \sum_{i\in J} N_i E_i$ and also we write $\Omega_Y^n = h^*\Omega_X^n + \sum_{i\in J}(\nu_i - 1)E_i$, where Ω^n stands for the sheaf of algebraic differential forms of degree n. (We use here the same notation for invertible sheaves and the corresponding divisors.) Let W be a complex algebraic variety. We denote $\mathrm{Eu}(W)$ its topological Euler-Poincaré characteristic with compact supports: $\mathrm{Eu}(W) := \sum_i (-1)^i \mathrm{rk}_{\mathbf{C}} H_c^i(W(\mathbf{C}))$. In fact, it can be shown, but we shall not use it, that $\mathrm{Eu}(W)$ is also equal to the topological Euler-Poincaré characteristic with compact supports $\sum_i (-1)^i \mathrm{rk}_{\mathbf{C}} H^i(W(\mathbf{C}))$.
Now, we can state the following result, which was first proved using p-adic integration in [14]:

2.1.1. **Theorem** (Denef-Loeser). *Let X be a smooth complex algebraic variety of dimension n and D a divisor on X. Then the rational function*

$$Z_{\mathrm{top}}(X,D)(s) := \sum_{I\subset J} \frac{\mathrm{Eu}(E_I^\circ)}{\prod_{i\in I}(N_i s + \nu_i)} \tag{2.1.1}$$

does not depend of the choice of a log-resolution $h : Y \to X$, but only of the pair (X, D).

Let us explain the idea of the proof. We shall assume, as in [14], that $X = \mathbf{A}^n$ and $D = f^{-1}(0)$, with f a polynomial in $\mathbf{C}[x_1, \ldots, x_n]$ but the proof in general works just the same. We shall write $Z_{\mathrm{top},f}(s)$ for $Z_{\mathrm{top}}(X,D)(s)$. Now, we shall make the assumption that the coefficients of f all lie in the same number field K, i.e. f is in $K[x_1, \ldots, x_n]$ (in general, we can only assume they lie in a field of finite type over $\mathbf{Q}$, but the basic idea of the proof still remains the same, see [14]).
Now for every prime ideal $\mathfrak{P}$ in the ring of integers $\mathcal{O}_K$, we denote by $K_{\mathfrak{P}}$ the corresponding local field, with ring of integers $\mathcal{O}_{\mathfrak{P}}$ and residue field $k_{\mathfrak{P}}$. We consider the local zeta function

$$Z_{f,K_{\mathfrak{P}}}(s) := \int_{\mathcal{O}_{\mathfrak{P}}^n} |f|_{\mathfrak{P}}^s |dx|_{\mathfrak{P}},$$

where $|\_|_{\mathfrak{P}}$ stands for the $\mathfrak{P}$-adic norm on $K_{\mathfrak{P}}$. Consider now a log-resolution $h : Y \to X$ defined over K. It follows from a formula of Denef [12] that, for almost all $\mathfrak{P}$,

$$Z_{f,K_{\mathfrak{P}}}(s) = q^{-n} \sum_{I\subset J} \mathrm{card}(E_I^\circ(k_{\mathfrak{P}})) \prod_{i\in I} \frac{(q-1)q^{-(N_i s+\nu_i)}}{1 - q^{-(N_i s+\nu_i)}}, \tag{2.1.2}$$

with $q = \mathrm{card} k_{\mathfrak{P}}$. Here we should explain what we mean by $\mathrm{card}(E_I^\circ(k_{\mathfrak{P}}))$. For Z a variety over K we choose a model $\mathcal{Z}$ over $\mathcal{O}_K$, i.e. a variety over $\mathcal{O}_K$ wich is isomorphic to Z over K, and we set $\mathrm{card}(X(k_{\mathfrak{P}})) = \mathrm{card}((\mathcal{X} \otimes k_{\mathfrak{P}})(k_{\mathfrak{P}}))$. Of course, this may depend on the choice of the model $\mathcal{X}$, but this will be the case only for a finite number of prime ideals $\mathfrak{P}$, so it makes sense to consider $\mathrm{card}(X(k_{\mathfrak{P}}))$ for almost all $\mathfrak{P}$. For $e \geq 1$, let us write $K_{\mathfrak{P}}^{(e)}$ for the unramified extension of $K_{\mathfrak{P}}$ of degree e. Its residue field $k_{\mathfrak{P}}^{(e)}$ has q^e elements. Also, for almost all $\mathfrak{P}$, equation (2.1.2) still holds when replacing $K_{\mathfrak{P}}$ by $K_{\mathfrak{P}}^{(e)}$, yielding

$$Z_{f,K_{\mathfrak{P}}^{(e)}}(s) = q^{-en} \sum_{I \subset J} \mathrm{card}(E_I^{\circ}(k_{\mathfrak{P}}^{(e)})) \prod_{i \in I} \frac{(q^e - 1)q^{-e(N_i s + \nu_i)}}{1 - q^{-e(N_i s + \nu_i)}}. \qquad (2.1.3)$$

Now, taking formally the limit as $e \mapsto 0$ in (2.1.3) gives us (2.1.1). This is quite clear, once we know that $\lim_{e \mapsto 0} \mathrm{card} W(k_{\mathfrak{P}}^{(e)}) = \mathrm{Eu} W(\mathbf{C})$, for almost all $\mathfrak{P}$, when W is a variety over K. This last fact follows from Grothendieck's trace formula for the Frobenius acting on ℓ-adic cohomology and standard comparison results between ℓ-adic and classical Betti cohomology. Indeed, we have $\mathrm{card} W(k_{\mathfrak{P}}^{(e)}) = \sum \alpha_i^e - \sum \beta_j^e$, with α_i and β_j the eigenvalues, respectively in even and odd degree, of the Frobenius acting on ℓ-adic cohomology groups with compact supports with compact supports, and taking $e = 0$ just gives the trace of the identity, i.e. the alternating sum of the ranks of ℓ-adic cohomology groups with compact supports. Of course, this is just a rough sketch of the proof and further work is required in order to show this process of taking limits as $e \mapsto 0$ really makes sense.

2.2. **Birational Calabi-Yau varieties have the same Betti numbers.** Let X be a smooth complex projective variety of dimension n. We say X is Calabi-Yau if X admits a nowhere vanishing degree n algebraic differential form ω. This is equivalent to the sheaf Ω_X^n being trivial. Recall the Betti numbers $b_i(X)$ are the ranks of the cohomology groups $H^i(X(\mathbf{C}), \mathbf{C})$. Considerations from theoretical physics (string theory) led to the guess that birational Calabi-Yau varieties should have the same Betti numbers (and even the same Hodge numbers, cf. 3.2.4).
This was proved by Batyrev [4] using p-adic integration and the Weil conjectures.

2.2.1. **Theorem** (Batyrev). *Let X and X' be complex Calabi-Yau varieties of dimension n. Assume X and X' are birationally equivalent. Then they have the same Betti numbers.*

Let us sketch the proof. For simplicity, we assume, as in the proof of Theorem 2.1.1 that X, X' and all the data are defined over some number field K (in general they are defined only over some field of finite type, but the basic idea of the proof is the same). We keep the notation of 2.1. By Hironaka there exists a smooth projective Y defined over K, and birational proper morphisms (also defined over K) $h : Y \to X$ and $h' : Y \to X'$. Furthermore we may assume there exists a divisor with normal crossings $E = \cup_{i \in J} E_i$ such that the exceptional locus of h and h' respectively, is a finite union of E_i's. We may write $\Omega_Y^n = h^* \Omega_X^n + \sum_{i \in J} (\nu_i - 1) E_i$ and $\Omega_Y^n = h'^* \Omega_{X'}^n + \sum_{i \in J} (\nu_i' - 1) E_i$. Since $h^* \Omega_X^n$ and $h'^* \Omega_{X'}^n$ are both trivial, it follows[1] that $\nu_i = \nu_i'$ for every i in J. One then deduces follows from the change of

[1] This is not completely evident, but can be proved quite easily using elementary algebraic geometry. Check it as an exercise.

variables formula, that for almost all $\mathfrak{P}$, with a slight abuse of notation, with have

$$\int_{X(K_{\mathfrak{P}})} |\omega|_{\mathfrak{P}} = \int_{X'(K_{\mathfrak{P}})} |\omega'|_{\mathfrak{P}}$$

and the same holds for all unramified extensions $K_{\mathfrak{P}}^{(e)}$. Indeed, we may express by the change of variables formula both integrals as the same integral over the rational points of Y. Since, for almost all $\mathfrak{P}$ and every e, $\int_{X(K_{\mathfrak{P}}^{(e)})} |\omega|_{\mathfrak{P}}$ is equal to $q^{-en}\mathrm{card}(X(k_{\mathfrak{P}}^{(e)}))$ (this is a special case of Denef's result above mentionned that goes back at least to A. Weil), it follows that for almost all $\mathfrak{P}$, the reductions of (some model of) X and X' mod $\mathcal{M}_{\mathfrak{P}}$ have same the zeta function. On the other side, for proper smooth varieties over a finite field, the zeta function determines the ℓ-adic Betti numbers by Deligne's proof of the Weil conjectures, hence the result follows from standard comparison results between ℓ-adic and usual Betti numbers.

2.2.2. *Remark.* The above proof gives in fact the following stronger result (see [4]): if X and X' are two n-dimensional smooth proper complex varieties that are K-equivalent, meaning that there exists birational proper morphisms $h : Y \to X$ and $h' : Y \to X'$ with Y smooth proper such that the invertible sheaves $h^*(\Omega^n_X)$ and $h^*(\Omega^n_{X'})$ are isomorphic, then X and X' have the same Betti numbers.

2.3. **Towards motivic integration.** The fact that one can use p-adic integration to prove results over $\mathbf{C}$ may look appealing to model theorists - after all using finite fields to prove surjectivity of injective polynomial complex morphisms goes back to Ax [3] - but it was challenging for geometers to find a more direct approach. Of course, one obvious try would like to perform some kind of integration over the field $\mathbf{C}((t))$, but since it is not locally compact it is hopeless to construct any reasonable real valued measureon it. The real breakthrough happened at the end of 1995, when Maxim Kontsevich got the idea of motivic integration: one should replace the real numbers by (the completion of) the Grothendieck ring of algebraic varieties as he explained in his seminal Orsay talk [35].

3. Additive invariants and Grothendieck rings

3.1. **Additive invariants of algebraic varieties.** Let R be a ring. We denote by Var_R the category of algebraic varieties over R, i.e. reduced and separated schemes of finite type over R. If X and X' are varieties over R, we denote by $X \times X'$ their cartesian product in Var_R. By definition $X \times X'$ is equal to $(X \otimes_{\mathrm{Spec} R} X')_{\mathrm{red}}$, that is the scheme $X \otimes_{\mathrm{Spec} R} X'$ endowed with its reduced structure. An additive invariant

$$\lambda : \mathrm{Var}_R \longrightarrow S,$$

with S a ring, assigns to any X in Var_R an element $\lambda(X)$ of S such that

$$\lambda(X) = \lambda(X')$$

for $X \simeq X'$,

$$\lambda(X) = \lambda(X') + \lambda(X \setminus X'),$$

for X' closed in X, and

$$\lambda(X \times X') = \lambda(X)\lambda(X')$$

for every X and X'.

Let us remark that additive invariants λ naturally extend to take their values on constructible subsets of algebraic varieties. Indeed a constructible subset W may be written as a finite disjoint union of locally closed subvarieties Z_i, $i \in I$. One may define $\lambda(W)$ to be $\sum_{i \in I} \lambda(Z_i)$. By the very axioms, this is independent of the decomposition into locally closed subvarieties.

3.2. **Examples.**

3.2.1. There exists a universal additive invariant $[\_] : \mathrm{Var}_R \to K_0(\mathrm{Var}_R)$ in the sense that composition with $[\_]$ gives a bijection between ring morphisms $K_0(\mathrm{Var}_R) \to S$ and additive invariants $\mathrm{Var}_R \to S$. The construction of $K_0(\mathrm{Var}_R)$ is quite easy: take the free abelian group on isomorphism classes $[X]$ of objects of Var_R and mod out by the relation $[X] = [X'] + [X \setminus X']$ for X' closed in X. The product is now defined by $[X][X'] = [X \times X']$.

We shall denote by $\mathbf{L}$ the class of the affine line $\mathbf{A}^1_R$ in $K_0(\mathrm{Var}_R)$. An important role will be played by the ring $\mathcal{M}_R := K_0(\mathrm{Var}_R)[\mathbf{L}^{-1}]$ obtained by localization with respect to the multiplicative set generated by $\mathbf{L}$. This construction is analogue to the construction of the category of Chow motives from the category of effective Chow motives by localization with respect to the Lefschetz motive. (Remark that the morphism χ_c of 3.2.5 sends $\mathbf{L}$ to the class of the Lefschetz motive.)

One should stress that very little is known about the structure of the rings $K_0(\mathrm{Var}_R)$ and $\mathcal{M}_R$ even when R is a field. Let us just quote a result by Poonen [40] saying that when k is a field of characteristic zero the ring $K_0(\mathrm{Var}_k)$ is not a domain. For instance, even for a field k, it is not known whether the localization morphism $K_0(\mathrm{Var}_k) \to \mathcal{M}_k$ is injective or not.

3.2.2. *Remark.* In fact, the ring $K_0(\mathrm{Var}_k)$ as well as the canonical morphism $\chi_c : K_0(\mathrm{Var}_k) \to K_0(\mathrm{CHMot}_k)$, were already considered by Grothendieck in a letter to Serre dated August 16, 1964, cf. p. 174 of [29].

3.2.3. *Euler characteristic.* Here $R = k$ is a field. When k is a subfield of $\mathbf{C}$, the Euler characteristic $\mathrm{Eu}(X) := \sum_i (-1)^i \mathrm{rk} H^i_c(X(\mathbf{C}), \mathbf{C})$ give rise to an additive invariant $\mathrm{Eu} : \mathrm{Var}_k \to \mathbf{Z}$. For general k, replacing Betti cohomology with compact support by ℓ-adic cohomology with compact support, $\ell \neq \mathrm{char} k$, one gets an additive invariant $\mathrm{Eu}_\ell : \mathrm{Var}_k \to \mathbf{Z}$, which does not depend on ℓ. Since the Euler number of the affine line is 1, the Euler characteristic extends to a morphism $\mathcal{M}_k \to \mathbf{Z}$.

3.2.4. *Hodge polynomial.* Let us assume $R = k$ is a field of characteristic zero. Then it follows from Deligne's Mixed Hodge Theory that there is a unique additive invariant $H : \mathrm{Var}_k \to \mathbf{Z}[u, v]$, which assigns to a smooth projective variety X over k its usual Hodge polynomial

$$H(u, v) := \sum_{p,q} (-1)^{p+q} h^{p,q}(X) u^p v^q,$$

with $h^{p,q}(X) = \dim H^q(X, \Omega^p_X)$ the (p,q)-Hodge number of X. This is is also a consequence of Bittner's Theorem that we shall explain in 3.3. Since $H(\mathbf{A}^1_k) = uv$, h extends to a ring morphism $\mathcal{M}_k \to \mathbf{Z}[u, v, (uv)^{-1}]$.

3.2.5. *Virtual motives.* More generally, when $R = k$ is a field of characteristic zero, there exists by Gillet and Soulé [25], Guillen and Navarro-Aznar [28], a unique additive invariant $\chi_c : \mathrm{Var}_k \to K_0(\mathrm{CHMot}_k)$, which assigns to a smooth projective variety X over k the class of its Chow motive, where $K_0(\mathrm{CHMot}_k)$ denotes the Grothendieck ring of the category of Chow motives over k (with rational coefficients). Let us explain what Chow motives and the category $K_0(\mathrm{CHMot}_k)$ are. Let $\mathcal{V}$ denote the category of smooth and projective $\mathbf{C}$-schemes. For an object X in $\mathcal{V}$ and an integer d, $\mathcal{Z}^d(X)$ denotes the free abelian group generated by irreducible subvarieties of X of codimension d. We define the rational Chow group $A^d(X)$ as the quotient of $\mathcal{Z}^d(X) \otimes \mathbf{Q}$ modulo rational equivalence. For X and Y in $\mathcal{V}$, we denote by $\mathrm{Corr}^r(X, Y)$ the group of correspondences of degree r from X to Y. If X is purely d-dimensional, $\mathrm{Corr}^r(X, Y) = A^{d+r}(X \times Y)$, and if $X = \coprod X_i$, $\mathrm{Corr}^r(X, Y) = \oplus \mathrm{Corr}^r(X_i, Y)$. The category Mot of $\mathbf{C}$-motives may be defined as follows (cf. [41]). Objects of Mot are triples (X, p, n) where X is in $\mathcal{V}$, p is an idempotent (i.e. $p^2 = p$) in $\mathrm{Corr}^0(X, X)$, and n is an integer in $\mathbf{Z}$. If (X, p, n) and (Y, q, m) are motives, then

$$\mathrm{Hom}_{\mathrm{Mot}}((X, p, n), (Y, q, m)) = q \, \mathrm{Corr}^{m-n}(X, Y) \, p.$$

Composition of morphisms is given by composition of correspondences. The category Mot is additive, $\mathbf{Q}$-linear, and pseudo-abelian. There is a natural tensor product on Mot, defined on objects by

$$(X, p, n) \otimes (Y, q, m) = (X \times Y, p \otimes q, n + m).$$

We denote by h the functor $h : \mathcal{V}^\circ \to \mathrm{Mot}$ which sends an object X to $h(X) = (X, \mathrm{id}, 0)$ and a morphism $f : Y \to X$ to its graph in $\mathrm{Corr}^0(X, Y)$. This functor is compatible with the tensor product and the unit motive $1 = h(\mathrm{Spec}\,\mathbf{C})$ is the identity for the product. We denote by $\mathbf{L}$ the Lefschetz motive $\mathbf{L} = (\mathrm{Spec}\,\mathbf{C}, \mathrm{id}, -1)$. One can prove there is a canonical isomorphism $h(\mathbf{P}^1) \simeq 1 \oplus \mathbf{L}$, so, in some sense, $\mathbf{L}$ corresponds to $H^2(\mathbf{P}^1)$.
Since algebraic correspondences naturally act on cohomology, any cohomology theory on the category $\mathcal{V}$ factors through CHMot_k hence motives have canonical Betti and Hodge realizations.

3.2.6. **Theorem.** *There exists a unique morphism of rings*

$$\chi_c : K_0(\mathrm{Var}_k) \longrightarrow K_0(\mathrm{CHMot}_k)$$

such that $\chi_c([X]) = [h(X)]$ *for* X *projective and smooth.*

Remark that $\chi_c([\mathbf{L}]) = \mathbf{L}$.

3.2.7. *Counting points.* Counting points also yields additive invariants. Assume $k = \mathbf{F}_q$, then $N_n : X \mapsto |X(\mathbf{F}_{q^n})|$ gives rise to an additive invariant $N_n : \mathrm{Var}_k \to \mathbf{Z}$. Similarly, if R is (essentially) of finite type over $\mathbf{Z}$, for every maximal ideal $\mathfrak{P}$ of R with finite residue field $k_{\mathfrak{P}}$, we have an additive invariant $N_{\mathfrak{P}} : \mathrm{Var}_R \to \mathbf{Z}$, which assigns to X the cardinality of $(X \otimes k_{\mathfrak{P}})(k_{\mathfrak{P}})$.

3.3. **Bittner's Theorem.** We assume form now on that k is a field of characteristic zero. It is a rather straightforward consequence of Hironaka's theorem that $K_0(\mathrm{Var}_k)$ is generated by classes of smooth irreducible projective varieties. More subtle is the following presentation by generators and relations of $K_0(\mathrm{Var}_k)$ due to F. Bittner [5]. We denote by $K_0^{\mathrm{bl}}(\mathrm{Var}_k)$ the quotient of the free abelian group on isomorphism classes of irreducible smooth projective varieties over k by the relations

$$[\mathrm{Bl}_Y X] - [E] = [X] - [Y],$$

for Y and X irreducible smooth projective over k, Y closed in X, $\mathrm{Bl}_Y X$ the blowup of X with center Y and E the exceptional divisor in $\mathrm{Bl}_Y X$. As for $K_0(\mathrm{Var}_k)$, cartesian product induces a product on $K_0^{\mathrm{bl}}(\mathrm{Var}_k)$ which endows it with a ring structure. There is a canonical ring morphism $K_0^{\mathrm{bl}}(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)$, which sends $[X]$ to $[X]$.

3.3.1**. Theorem** (Bittner [5])**.** *Assume k is of characteristic zero. The canonical ring morphism*

$$K_0^{\mathrm{bl}}(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)$$

is an isomorphism.

The proof is based on Hironaka resolution of singularities and the weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk [1] which we quote in the following version:

3.3.2**. Theorem** (Weak factorization theorem)**.** *Let k be a field of characteristic zero. Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between proper smooth irreducible varieties over k. Let $U \subset X_1$ be the largest open subset on which ϕ is an isomorphism. Then ϕ can be factored into a sequence of blowing ups and blowing down with smooth centers disjoint from U: $\phi_i : V_{i-1} \dashrightarrow V_i$, $i = 1, \dots, \ell$, with $V_0 = X_1$, $V_\ell = X_2$, with ϕ_i or ϕ_i^{-1} blowing ups with smooth centers away from U. Moreover there exists i_0 such that $V_i \dashrightarrow X_1$ is defined everywhere and projective for $i \leq i_0$ and $V_i \dashrightarrow X_2$ is defined everywhere and projective for $i \geq i_0$.*

Theorem 3.3.1 is a very efficient tool to provide additive invariants. Indeed, it is enough to know the invariant for smooth projective varieties and to check it behaves properly for blowing ups with smooth centers. In particular it is now a straightforward consequence of theorem 3.3.1 (but using the full strenght of weak factorization) that the Hodge-Deligne polynomial of 3.2.4 and the virtual motives of 3.2.5 are well-defined additive invariants.

3.4. **Grothendieck rings of first order theories.** The Grothendieck ring $K_0(\mathrm{Var}_k)$ may be generalized as follows to any first order theory. Let $\mathcal{L}$ be a first order language and let T be a theory in the language $\mathcal{L}$.
We denote by $K_0(T)$ the quotient of the free abelian group generated by symbols $[\varphi]$ for φ a formula in $\mathcal{L}$ by the subgroup generated by the following relations

(1) If φ is a formula in $\mathcal{L}$ with free variables $x = (x_1, \ldots, x_n)$ and φ' is a formula in $\mathcal{L}$ with free variables $x' = (x'_1, \ldots, x'_{n'})$, then $[\varphi] = [\varphi']$ if there exists a formula ψ in $\mathcal{L}$, with free variables (x, x'), such that

$$T \models [\forall x(\varphi(x) \to \exists! x' : (\varphi'(x') \wedge \psi(x, x')))] \wedge [\forall x'(\varphi'(x') \to \exists! x : (\varphi(x) \wedge \psi(x, x')))].$$

(2) $[\varphi \vee \varphi'] = [\varphi] + [\varphi'] - [\varphi \wedge \varphi']$, for φ and φ' formulas in $\mathcal{L}$.

Furthermore one puts a ring structure on $K_0(T)$ by setting

(3) $[\varphi(x)] \cdot [\varphi'(x')] = [\varphi(x) \wedge \varphi'(x')]$, if φ and φ' are formulas in $\mathcal{L}$ with disjoint free variables x and x'.

For every interpretation of a theory T_1 in a theory T_2 there is a canonical morphism of rings $K_0(T_1) \to K_0(T_2)$, and this gives rise to a functor from the category of theories in $\mathcal{L}$, morphisms being given by interpretation, to the category of commutative rings.

If k is a field and T_{ac} is the theory of algebraically closed fields containing k, then $K_0(T_{\mathrm{ac}})$ is isomorphic to $K_0(\mathrm{Var}_k)$. If $T_{\mathbf{R}}$ is the theory of real closed fields in the language of ordered rings, then $K_0(T_{\mathbf{R}})$ is isomorphic to $\mathbf{Z}$. Recently, Cluckers and Haskell [7] proved that the theory of any fixed p-adic field, in the language of rings, has trivial Grothendieck group. In fact, Cluckers proved in [8] that for any two p-adic semi-algebraic X and X' sets of the same dimension $d > 0$, there exists a semi-algebraic isomorphism between X and X'.

4. Geometric motivic integration

4.1. **Arc spaces.** Arc spaces are the $k[[t]]$-analogue of p-adic points. Let k be a field of characteristic 0. Many of the results presented in these lectures do not hold anymore or become unknown in positive characteristic.
For $n \geq 0$, we introduce the space of n-arcs on X, denoted by $\mathcal{L}_n(X)$. This is an algebraic variety which represents the functor:

$$k - \text{algebras} \longrightarrow \text{Sets}$$

$$R \mapsto \mathrm{Hom}_{k-schemes}(\mathrm{Spec}(R[t]/(t^{n+1}), X) := X(R[t]/(t^{n+1})).$$

For example when X is an affine variety with equations $f_i(\vec{x}) = 0$, $i = 1, \cdots, m$, $\vec{x} = (x_1, \cdots, x_r)$, then $\mathcal{L}_n(X)$ is given by the equations, in the variables $\vec{a}_0, \cdots, \vec{a}_n$, expressing that $f_i(\vec{a}_0 + \vec{a}_1 t + \cdots + \vec{a}_n t^n) \equiv 0 \mod t^{n+1}, i = 1, \cdots, m$.
We have canonical isomorphisms $\mathcal{L}_0(X) = X$ and $\mathcal{L}_1(X) = TX$, where TX denotes the tangent space of the variety X.
For $m \geq n$ there are canonical morphisms $\theta^n_m : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$. In general, when X is not smooth, they need nor to be surjective. When X is smooth of dimension d, θ^n_m is a locally trivial fibration for the Zariski topology with fiber $\mathbf{A}^{(m-n)d}$.

Taking the projective limit of these algebraic varieties $\mathcal{L}_n(X)$ we obtain the arc space $\mathcal{L}(X)$ of X. A priori this is just a pro-scheme, but since the transition maps θ_m^n are affine it is indeed a k-scheme.
In general, $\mathcal{L}(X)$ is not of finite type over k. The K-rational points of $\mathcal{L}(X)$ are the $K[[t]]$-rational points of X. These are called K-arcs on X. For example when X is an affine variety with equations $f_i(\vec{x}) = 0, i = 1, \cdots m, \vec{x} = (x_1, \cdots, x_r)$, then the K-rational points of $\mathcal{L}(X)$ are the sequences $(\vec{a}_0, \vec{a}_1, \vec{a}_2, \cdots) \in (K^n)^{\mathbf{N}}$ satisfying $f_i(\vec{a}_0 + \vec{a}_1 t + \vec{a}_2 t^2 + \cdots) = 0$, for $i = 1, \cdots, m$. For every n we have natural morphisms

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$$

obtained by truncation. For any arc γ on X (i.e. a K-arc for some field K containing k), we call $\pi_0(\gamma)$ the origin of the arc γ.
One can easily check that $\mathcal{L}(X)$ represents the functor

$$k - \text{algebras} \longrightarrow \text{Sets}$$

$$R \mapsto \text{Hom}_{k-schemes}(\text{Spec}(R[[t]]), X) := X(R[[t]]).$$

It also represents the functor

$$k - \text{schemes} \longrightarrow \text{Sets}$$

$$S \mapsto \text{Hom}_{\text{locally ringed spaces}}((S, \mathcal{O}_S[[t]]), X).$$

If $f : Y \to X$ is a morphism of varieties, we shall still denote by f the corresponding morphism $\mathcal{L}(Y) \to \mathcal{L}(X)$.

4.2. **Completing $\mathcal{M}_k$.** We want to assign a measure to subsets of $\mathcal{L}(X)$. This measure will take values in a ring related to $K_0(\text{Var}_k)$. In the analogy with p-adic integration, $K_0(\text{Var}_k)$ is the analogue of $\mathbf{Z}$ and $\mathcal{M}_k$ is the analogue of $\mathbf{Z}[p^{-1}]$ (the number of rational points of the affine line over $\mathbf{F}_p$ is p). Since in $\mathbf{R}$, $p^{-n} \mapsto 0$ as $n \mapsto \infty$, we shall complete $\mathcal{M}_k$ is such a way that $\mathbf{L}^{-n} \mapsto 0$ as $n \mapsto \infty$. This is achieved in the following way: we define $F^m\mathcal{M}_k$ to be the subgroup of $\mathcal{M}_k$ generated by elements of the form $[S]\mathbf{L}^{-i}$, with $\text{dim}S - i \leq -m$. We have $F^{m+1} \subset F^m$, $\mathbf{L}^{-m} \in F^m$ and $F^nF^m \subset F^{n+m}$. We denote by $\widehat{\mathcal{M}}_k$ the completion of $\mathcal{M}_k$ with respect to that filtration.
A minor technical issue shows up here, since it is not known whether the canonical morphism $\mathcal{M}_k \to \widehat{\mathcal{M}}_k$ is injective or not. Nevertheless, this is not too much a problem for applications by the following:

4.2.1. **Proposition.** *Invariants* Eu : $\mathcal{M}_k \to \mathbf{Z}$ *(Euler number) and* $H : \mathcal{M}_k \to \mathbf{Z}[u, v, (uv)^{-1}]$ *(Hodge polynomial) factor through the image* $\overline{\mathcal{M}}_k$ *of* $\mathcal{M}_k$ *in* $\widehat{\mathcal{M}}_k$.

Proof. Since Eu $= H(1, 1)$, it is enough to prove the result for H. But if a is in $F^m\mathcal{M}_k$, the total degree of $h(a)$ is $\leq -2m$, so if a belongs to the kernel $\cap_m F^m\mathcal{M}_k$ of $\mathcal{M}_k \to \widehat{\mathcal{M}}_k$, $h(a)$ should be zero. □

4.3. **Measurable sets.** For more details about this section, see the appendix to [18]. Let X be an algebraic variety over k of dimension d, maybe singular. By a cylinder in $\mathcal{L}(X)$, we mean a subset A of $\mathcal{L}(X)$ of the form $A = \pi_n^{-1}(C)$ with C a constructible subset of $\mathcal{L}_n(X)$, for some n. We say A is stable (at level n) if furthermore $\pi_{m+1}(\mathcal{L}(X)) \to \pi_m(\mathcal{L}(X))$ is a piecewise Zariski fibration over $\pi_m(A)$ with fiber $\mathbf{A}_k^d$ for all $m \geq n$. By a piecewise Zariski fibration over $\pi_m(A)$ we mean that there exists a finite partition of $\pi_m(A)$ into locally closed subsets of $\mathcal{L}_m(X)$ over which the morphism is a locally trivial fibration for the Zariski topology.
If A is a stable cylinder at level n, we set

$$\tilde{\mu}(A) := [\pi_n(A)]\mathbf{L}^{-(n+1)d}$$

in $\mathcal{M}_k$. Remark that the stability condition insures that we would get the same value by viewing A as a stable cylinder at level m, $m \geq n$. Also, it can be proved that if X is smooth, all cylinders are stable. In particular, in this case $\mathcal{L}(X)$ itself is a stable cylinder and $\tilde{\mu}(\mathcal{L}(X)) = [X]\mathbf{L}^{-d}$.
In general, we can assign to any cylinder A in $\mathcal{L}(X)$ a measure $\mu(A)$ in $\widehat{\mathcal{M}}_k$ by a limit process as follows: for $e \geq 0$, set $\mathcal{L}^{(e)}(X) := \mathcal{L}(X) \setminus \pi_e^{-1}(\pi_e(\mathcal{L}(X_{\mathrm{sing}})))$, where X_{sing} denote the singular locus of X and we view $\mathcal{L}(X_{\mathrm{sing}})$ as a subset of $\mathcal{L}(X)$. The set $\mathcal{L}^{(e)}(X)$ should be viewed as $\mathcal{L}(X)$ minus some tubular neighborhood of the singular locus. It can be proved that $A \cap \mathcal{L}^{(e)}(X)$ is a stable cylinder and that $\tilde{\mu}(A \cap \mathcal{L}^{(e)}(X))$ does have a limit in $\widehat{\mathcal{M}}_k$ as e goes to ∞. We denote this limit by $\mu(A)$. This apply in particular to $A = \mathcal{L}(X)$ when X is not smooth.
We shall define

$$||\_|| : \widehat{\mathcal{M}}_k :\to \mathbf{R}_{\geq 0}$$

to be given by $||a|| = 2^{-n}$ if $a \in F^n\widehat{\mathcal{M}}_k$ and $a \notin F^{n+1}\widehat{\mathcal{M}}_k$, where $F^\bullet\widehat{\mathcal{M}}_k$ denotes the induced filtration on $\widehat{\mathcal{M}}_k$.
We shall say a subset A of $\mathcal{L}(X)$ is measurable if, for every $\varepsilon > 0$, there exists cylinders $A_i(\varepsilon)$, $i \in \mathbf{N}$, such that $(A \cup A_0(\varepsilon)) \setminus (A \cap A_0(\varepsilon))$ is contained in $\cup_{i\geq 1} A_i(\varepsilon)$, and $||\mu(A_i(\varepsilon))|| \leq \varepsilon$, for every $i \geq 1$. Then one can show (cf. appendix to [18]) that $\mu(A) := \lim_{\varepsilon \mapsto 0} \mu(A_0(\varepsilon))$ exists and is independent of the choice of the $A_i(\varepsilon)$'s. We say that A is strongly measurable if moreover we can take $A_0(\varepsilon) \subset A$.
Let A be a measurable subset of $\mathcal{L}(X)$ and $\alpha : A \to \mathbf{Z} \cup \{\infty\}$ be a function such that all its fibers are measurable. We shall say $\mathbf{L}^\alpha$ is integrable if the series

$$\int_A \mathbf{L}^{-\alpha} d\mu := \sum_{n \in \mathbf{Z}} \mu(A \cap \alpha^{-1}(n))\mathbf{L}^{-n}$$

is convergent in $\widehat{\mathcal{M}}_k$.

4.4. **Semi-algebraic subsets.** An important class of measurable sets is that of semi-algebraic subsets of $\mathcal{L}(X)$. We shall explain here only what are semi-algebraic subsets of $\mathcal{L}(\mathbf{A}_k^n)$, the definition for general X being deduced by using charts from the affine case.

We shall view points of $\mathcal{L}(\mathbf{A}_k^n)$ as n-uplets of formal power series. A semi-algebraic subset of $\mathcal{L}(\mathbf{A}_k^n)$ is a finite boolean combination of subsets defined by conditions of the form

$$\text{ord} f_1(x_1,\ldots,x_m) \geq \text{ord} f_2(x_1,\ldots,x_m) + L(\ell_1,\ldots,\ell_r) \tag{1}$$

$$\text{ord} f_1(x_1,\ldots,x_m) \equiv L(\ell_1,\ldots,\ell_r) \pmod{d} \tag{2}$$

and

$$h(\text{ac}(f_1(x_1,\ldots,x_m)),\ldots,\text{ac}(f_{m'}(x_1,\ldots,x_m))) = 0, \tag{3}$$

where f_i are polynomials with coefficients in $k[[t]]$, h is a polynomial with coefficients in k, L is a polynomial of degree ≤ 1 over $\mathbf{Z}$, $d \in \mathbf{N}$, and $\text{ac}(x)$ is the coefficient of lowest degree in t of x if $x \neq 0$, and is equal to 0 otherwise. Here we use the convention that $\infty + \ell = \infty$ and $\infty \equiv \ell \mod d$, for all $\ell \in \mathbf{Z}$. In particular the algebraic condition $f(x_1,\ldots,x_m) = 0$, for f a polynomial over $k[[t]]$, defines a semi-algebraic subset.
The following (consequence of a) quantifier elimination Theorem of J. Pas [39] is of fundamental use in the theory:

4.4.1. **Theorem.** *Let $\pi : \mathbf{A}_k^n \to \mathbf{A}_k^{n-1}$ be the projection on the $n-1$ first coordinates. If A is a semi-algebraic subset of $\mathcal{L}(\mathbf{A}_k^n)$, then $\pi(A)$ is a semi-algebraic subset of $\mathcal{L}(\mathbf{A}_k^{n-1})$.*

One can prove that every semi-algebraic subset of $\mathcal{L}(X)$ is strongly measurable. Furthermore, we have the following nice description of $\mu(A)$ in this case, which is an analogue of a p-adic result of Oesterlé [38], cf. [16]:

4.4.2. **Theorem** (Denef-Loeser). *If A is a semi-algebraic subset of $\mathcal{L}(X)$, with X of dimension d, then $\mu(A)$ is equal to limit of $[\pi_n(A)]\mathbf{L}^{-(n+1)d}$ in $\widehat{\mathcal{M}}_k$.*

Note that $[\pi_n(A)]$ in the above statement makes sense since one can deduce from Pas' Theorem that $\pi_n(A)$ is constructible.

4.5. **Pas quantifier elimination.** We give here the original statement of Pas quantifier elimination. This subsection may be skipped at first reading but it will be used later in the lectures. Let K be a valued field, with valuation $\text{ord} : K \to \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group. We denote by $\mathcal{O}_K$ the valuation ring, by $\mathcal{M}_K$ the valuation ideal, by U the group of units in $\mathcal{O}_K$, by k the residue field, and by $\text{Res} : \mathcal{O}_K \to k$ the canonical projection. We assume that K has an angular component map. By this we mean a map $\text{ac} : K \to k$ such that $\text{ac}0 = 0$, the restriction of ac to $K^\times$ is multiplicative and the restriction of ac to U coincides with the restriction of Res. From now on we fix that angular component map ac.
We consider 3-sorted first order languages of the form

$$\mathcal{L} = (\mathbf{L}_K, \mathbf{L}_k, \mathbf{L}_\Gamma, \text{ord}, \text{ac}),$$

consisting of

(i) the language $\mathbf{L}_K = \{+, -, \times, 0, 1\}$ of rings as valued field sort,

(ii) the language $\mathbf{L}_k = \{+, -, \times, 0, 1\}$ of rings as residue field sort,
(iii) a language $\mathbf{L}_\Gamma$, which is an extension of the language $\{+, 0, \infty, \leq\}$ of ordered abelian groups with an element ∞ on top, as the value sort,
(iv) a function symbol ord from the valued field sort to the value sort, which stands for the valuation,
(v) a function symbol ac from the valued field sort to the residue field sort, which stands for the angular component map.

In the following we shall assume that K is henselian and that k is of characteristic zero. We consider $(K, k, \Gamma \cup \{\infty\}, \mathrm{ord}, \mathrm{ac})$ as a structure for the language $\mathcal{L}$, the interpretations of symbols being the standard ones. By an henselian $\mathcal{L}$-extension of K, we mean an extension $(K', k', \Gamma' \cup \{\infty\}, \mathrm{ord}', \mathrm{ac}')$ of the structure $(K, k, \Gamma \cup \{\infty\}, \mathrm{ord}, \mathrm{ac})$ with respect to the language $\mathcal{L}$, with K' a henselian valued field. (By an extension, we mean a structure for the language $\mathcal{L}$ which contains the original structure as a substructure.) By abuse of language we shall say that K' is a henselian $\mathcal{L}$-extension of K.
We may now state the quantifier elimination Theorem of Pas [39].

4.5.1. **Theorem** (Pas). *Let K be a valued field which satisfies the previous conditions. For every $\mathcal{L}$-formula φ there exists an $\mathcal{L}$-formula φ' without quantifiers over the valued field sort, such that φ is equivalent in K' to φ', for every henselian $\mathcal{L}$-extension K' of K.*

In particular, when the value group is $\mathbf{Z}$, we shall use the language

$$\mathcal{L}_{\mathrm{Pas}} = (\mathbf{L}_K, \mathbf{L}_k, \mathcal{L}_{\mathrm{PR}\infty}, \mathrm{ord}, \mathrm{ac}),$$

where $\mathcal{L}_{\mathrm{Pres}\infty} = \mathcal{L}_{\mathrm{Pres}} \cup \{\infty\}$, whith $\mathcal{L}_{\mathrm{Pres}}$ the Presburger language.

4.6. **Change of variables formula.** We have the following motivic analogue of the p-adic change of variables formula 1.2.1:

4.6.1. **Theorem** (Change of variables formula). *Let X be an algebraic variety over k of dimension d. Let $h : Y \to X$ be a proper birational morphism. We assume Y to be smooth. Let A be a subset of $\mathcal{L}(X)$ such that A and $h^{-1}(A)$ are strongly measurable. Assume $\mathbf{L}^{-\alpha}$ is integrable on A. Then*

$$\int_A \mathbf{L}^{-\alpha} d\mu = \int_{h^{-1}(A)} \mathbf{L}^{-\alpha \circ h - \mathrm{ord} h^*(\Omega^d_X)} d\mu.$$

We should explain what is meant by $\mathrm{ord} h^*(\Omega^d_X)$. Firstly, if $\mathcal{I}$ is some ideal sheaf on Y, we denote by $\mathrm{ord}\mathcal{I}$ the function which to a arc φ in $\mathcal{L}(Y)$ assigns $\inf \mathrm{ord} g(\varphi)$ where g runs over local sections of $\mathcal{I}$ at $\pi_0(\varphi)$. We set Ω^d_X to be the d-th exterior power of Ω^1_X, the Kähler differentials. The image of $h^*(\Omega^d_X)$ in Ω^d_Y is of the form $\mathcal{I}\Omega^d_Y$ and we set $\mathrm{ord} h^*(\Omega^d_X) := \mathrm{ord}\mathcal{I}$.
The proof of Theorem 4.6.1, given in section 3 of [16] (for A semi-algebraic), relies on the following geometric statement (Lemma 3.4 of [16]):

4.6.2. **Proposition** (Denef-Loeser). *Let X be an algebraic variety over k. Let $h : Y \to X$ be proper birational morphism. We assume Y to be smooth. For e and e' in $\mathbf{N}$, we set*

$$\Delta_{e,e'} := \Big\{ \varphi \in \mathcal{L}(Y) \,\Big|\, \mathrm{ord}^{h} * (\Omega^d_X)(\varphi) = e \quad and \quad h(\varphi) \in \mathcal{L}^{(e)}(X) \Big\}.$$

Then there exists $c > 0$ such that, for $n \geq \sup(2e, e + ce')$,

(1) *The image $\Delta_{e,e',n}$ of $\Delta_{e,e'}$ in $\mathcal{L}_n(Y)$ is a union of fibers of h_n, the morphism induced by h.*
(2) *The morphism $h_n : \Delta_{e,e',n} \to h_n(\Delta_{e,e',n})$ is a piecewise Zariski fibration with fiber $\mathbf{A}^e_k$.*

4.7. **Some applications.** We can now reprove and reinterpret the results in section 2 using motivic integration.
Let us begin by Batyrev's Theorem 2.2.1.

4.7.1. **Theorem** (Kontsevich). *Let X and X' be two proper smooth varieties over k. Assume there are K-equivalent, i.e. that there exists birational proper morphisms $h : Y \to X$ and $h' : Y \to X'$ with Y smooth proper such that the invertible sheaves $h^*(\Omega^n_X)$ and $h^*(\Omega^n_{X'})$ are isomorphic. The $[X] = [X']$ in $\overline{\mathcal{M}}_k$.*

Proof. Since $\int_{\mathcal{L}(Y)} \mathbf{L}^{-\mathrm{ord} h^*(\Omega^d_X)} d\mu = \int_{\mathcal{L}(Y)} \mathbf{L}^{-\mathrm{ord} h'^*(\Omega^d_{X'})} d\mu$, it follows from the change of variables formula that $\mu(\mathcal{L}(X)) = \mu(\mathcal{L}(X'))$, hence $[X] = [X']$, since X and X' are smooth. □

4.7.2. **Corollary.** *Let X and X' be two proper smooth varieties over k. Assume there are K-equivalent (this holds in particular if they are both Calabi-Yau). Then they have the same Hodge numbers and Betti numbers.*

4.7.3. *Remark.* Theorem 4.7.1 can also be proved as a consequence of the weak factorization Theorem 3.3.2.

Now we shall give an intrisic meaning to the topological zeta function. Let X be a smooth algebraic variety of dimension n over k and D a divisor on X. For any integer $m \geq 0$ we may consider $\mathcal{X}_n := \{\varphi \in \mathcal{L}_m(X) | \mathrm{ord} f(\varphi) = m\}$, whet f is a local equation for d at $\pi_0(\varphi)$. We consider $Z_{\mathrm{mot,naive}}(X, D)(T)$ to be the formal series

$$Z_{\mathrm{mot,naive}}(X, D)(T) := \sum_{m \in \mathbf{N}} [\mathcal{X}_n] \mathbf{L}^{-mn} T^m$$

in $\mathcal{M}_k[[T]]$. This is the naive motivic zeta function attached to the pair (X, D). One deduces from Proposition 4.6.2 the following formula for $Z_{\mathrm{mot,naive}}(X, D)(T)$ in terms of a log-resolution of (X, D).

4.7.4. **Proposition** (Denef-Loeser). *Let $h : Y \to X$ be a log-resolution of (X, D). With the notations of 2.1 we have*

$$Z_{\mathrm{mot,naive}}(X, D)(T) = \mathbf{L}^{-n} \sum_{I \subset J} [E^\circ_I] \prod_{i \in I} \frac{(\mathbf{L} - 1)\mathbf{L}^{-\nu_i} T^{N_i}}{1 - \mathbf{L}^{-\nu_i} T^{N_i}}.$$

Let us formally evaluate $Z_{\mathrm{mot,naive}}(X,D)(T)$ at $T=\mathbf{L}^{-s}$ for s an integer ≥ 1. We obtain from the above Proposition a well defined element

$$\mathbf{L}^{-n}\sum_{I\subset J}[E_I^\circ]\prod_{i\in I}\frac{(\mathbf{L}-1)\mathbf{L}^{-(N_is+\nu_i)}}{1-\mathbf{L}^{-(N_is+\nu_i)}}.$$

in the ring $\mathcal{M}_{k,\mathrm{loc}}$ obtained from $\mathcal{M}_k$ by inverting the the elements $[\mathbf{P}^i_k]=1+\mathbf{L}+\mathbf{L}^2+\cdots+\mathbf{L}^i$, for $i=1,2,3,\cdots$, where $\mathbf{P}^i_k$ denotes the i-dimensional projective space over k. The ring morphism $\mathrm{Eu}:\mathcal{M}_k\to\mathbf{Z}$ extends uniquely to a ring morphism $\mathrm{Eu}:\mathcal{M}_{k,\mathrm{loc}}\to\mathbf{Q}$.
Hence we deduce the following conceptual and intrinsic interpretation of $Z_{\mathrm{top}}(X,D)(s)$:

4.7.5. **Proposition** (Denef-Loeser). *For every integer $s\geq 1$,*

$$Z_{\mathrm{top}}(X,D)(s)=\mathrm{Eu}(Z_{\mathrm{mot,naive}}(X,D)(\mathbf{L}^{-s})). \tag{4.7.1}$$

4.7.6. *Remark.* One may also prove Theorem 2.1.1 by using the weak factorization Theorem 3.3.2 (which was not available at the time of the first two proofs), but then one would miss the intrinsic interpretation (4.7.1).

Another important feature of $Z_{\mathrm{mot,naive}}$ is that it "contains" also the corresponding p-adic integrals. More precisely:

4.7.7. **Proposition** (Denef-Loeser). *Let K be a number field, set $X=\mathbf{A}^n_K$ and let D be the divisor of a polynomial f in $K[x_1,\dots,x_n]$. For almost all $\mathfrak{P}$,*

$$N_{\mathfrak{P}}(Z_{\mathrm{mot,naive}}(X,D)(\mathbf{L}^{-s}))$$

is equal to the p-adic integral

$$Z_{f,K_{\mathfrak{P}}}(s):=\int_{\mathcal{O}^n_{\mathfrak{P}}}|f|^s_{\mathfrak{P}}|dx|_{\mathfrak{P}}.$$

Here, we extend $N_{\mathfrak{P}}$ to $\mathcal{M}_K(\mathbf{L}^{-s})$ by sending $\mathbf{L}^{-s}$ to q^{-s}, with q the cardinal of $k_{\mathfrak{P}}$.

Proof. Follows directly from (2.1.2) and Proposition 4.7.4. □

4.8. **Geometrization of Q.** Let us now slighly generalize the Poincaré series Q and P. For X a variety over a $\mathcal{O}_K$, for K a finite extension of $\mathbf{Q}_p$. We set $N_m:=|X(\mathcal{O}_K/\varpi^{m+1})|$, for $m\geq 0$ and consider the series $Q(T):=\sum_{m\geq 0}N_mT^m$. The series $Q(T)$ is still rational (cf. [37], see also the review MR 83g:12015). Also, we denote by $\tilde{N}_m$ the cardinality of the image of $X(\mathcal{O}_K)$ in $X(\mathcal{O}_K/\varpi^{m+1})$. In other words, $\tilde{N}_m$ is the number of points in $X(\mathcal{O}_K/\varpi^{m+1})$ that may be lifted to actual points in $X(\mathcal{O}_K)$ and we set $P(T):=\sum_{m\geq 0}\tilde{N}_mT^m$. Denef's rationality proof extends to this setting. When X is defined by $f=0$ in the affine space one recovers the previous definitions.
It follows from Proposition 4.7.7 that, when X is an hypersurface in the affine space defined over some number field K, $N_{\mathfrak{P}}(Q_{\mathrm{geom}}(T))=Q_{X\otimes\mathcal{O}_{K_{\mathfrak{P}}}}(T)$. forr almost all $\mathfrak{P}$. This may be extended to any X over a number field K (cf. [20]).

4.9. **Geometrization of P.** As a geometric analogue of the Serre series $P(T)$, it is natural to consider, for a variety X over a field k, the generating series

$$P_{\mathrm{geom}}(T) := \sum_{m \geq 0} [\pi_m(\mathcal{L}(X))] \, T^m$$

in $\mathcal{M}_k[[T]]$. Here we should check that the image $\pi_m(\mathcal{L}(X))$ of $\mathcal{L}(X)$ in $\mathcal{L}_m(X)$ is a constructible subset of $\mathcal{L}_m(X)$. This holds thanks to Greenberg's Theorem on solutions of polynomial systems in Henselian rings [27], which states that $\pi_m(\mathcal{L}(X))$ is equal to $\theta_n^m(\mathcal{L}_n(X))$ for some $n \geq m$, together with Chevalley's constructibility Theorem.

4.9.1. **Theorem** (Denef-Loeser [16]). *Assume* $\mathrm{char} k = 0$. *The series* $P_{\mathrm{geom}}(T)$ *in* $\mathcal{M}_k[[T]]$ *is rational of the form*

$$\frac{R(T)}{\prod(1 - \mathbf{L}^a T^b)},$$

with $R(T)$ *in* $\mathcal{M}_k[T]$, a *in* $\mathbf{Z}$ *and* b *in* $\mathbf{N} \setminus \{0\}$.

The prove of this result follows similar lines than the proof of Theorem 1.4.1, using motivic integration instead of p-adic integration and Pas' quantifier elimination instead of Macintyre's.

4.10. **Towards P_{ar}.** When X is defined over a number field K, a quite natural guess would be, by analogy with what we have seen so far, that, for almost all finite places $\mathfrak{P}$, $N_{\mathfrak{P}}(P_{\mathrm{geom}}(T)) = P_{X \otimes \mathcal{O}_{K_{\mathfrak{P}}}}(T)$. But such a statement cannot hold true as can be seen on some simple examples. This is due to the fact that, in the very definition of $P(T)$, one is concerned in not considering extensions of the residue field, while in the definition of $P_{\mathrm{geom}}(T)$ extensions of the residue field k are allowed. To remedy this, one needs to be more careful about rationality issues concerning the residue field, and this will be the topic of the next section.

5. Assigning virtual Chow motives to formulas

5.1. **Subassignments.** Fix a ring R. We denote by Field_R the category of R-algebras that are fields. For an R-scheme X, we denote by h_X the functor which to a field K in Field_R assigns the set $h_X(K) := X(K)$. By a subassignment $h \subset h_X$ of h_X we mean the datum, for every field K in Field_R, of a subset $h(K)$ of $h_X(K)$. We stress that, contrarly to subfunctors, no compatibility is required between the various sets $h(K)$.

All set theoretic constructions generalize in an obvious way to the case of subassignments. For instance if h and h' are subassignments of h_X, then we denote by $h \cap h'$ the subassignment $K \mapsto h(K) \cap h'(K)$, etc.

Also, if $\pi : X \to Y$ is a morphism of R-schemes and h is a subassignment of h_X, we define the subassignment $\pi(h)$ of h_Y by $\pi(h)(K) := \pi(h(K)) \subset h_Y(K)$.

5.2. **Definable subassignments.** Let R be a ring. By a ring formula φ over R, we mean a first order formula in the language of rings with coefficients in R and free variables $x_1, \dots x_n$.
To a ring formula φ over R with free variables $x_1, \dots x_n$ one assigns the subassignment h_φ of $h_{\mathbf{A}^n_R}$ defined by
(5.2.1)

$$h_\varphi(K) := \Big\{(a_1, \dots, a_n) \in K^n \,\Big|\, \varphi(a_1, \dots, a_n) \text{ holds in } K\Big\} \subset K^n = h_{\mathbf{A}^n_R}(K).$$

Such a subassignment of $h_{\mathbf{A}^n_R}$ is called a definable subassignment. More generally, using affine coverings, cf. [17], one defines definable subassignments of h_X for X a variety over R.
It is quite easy to show that if $\pi : X \to Y$ is an R-morphism of finite presentation, $\pi(h)$ is a definable subassignment of h_Y if h is a definable subassignment of h_X.
In our situation, we are concerned with the subassignment $\pi(h_{\mathcal{L}(X)}) \subset h_{\mathcal{L}_n(X)}$. Remark that $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$ is not of finite type. Nevertheless, it follows from Pas' Theorem (cf. Proposition 6.2.2) that $\pi(h_{\mathcal{L}(X)})$ is a definable subassignment of $h_{\mathcal{L}_n(X)}$.

5.3. **Pseudo-finite fields.** Let φ be a formula over a number field K. For almost all finite places $\mathfrak{P}$ with residue field $k(\mathfrak{P})$, one may extend the definition in (5.2.1) to give a meaning to $h_\varphi(k(\mathfrak{P}))$. If φ and φ' are formulas over K, we set $\varphi \equiv \varphi'$ if $h_\varphi(k(\mathfrak{P})) = h_{\varphi'}(k(\mathfrak{P}))$ for almost all finite places $\mathfrak{P}$.
It follows from a fundamental result of J. Ax [2] that $\varphi \equiv \varphi'$ if and only if $h_\varphi(L) = h_\varphi(L')$ for every pseudo-finite field L containing K. We recall that a pseudo-finite field L is a perfect infinite field which has exactly one extension of each degree and such that every absolutely irreducible variety over L has a rational point over L. J. Ax proved [2] that an infinite field F is pseudo-finite if and only if every sentence in the first order language of rings which is true in all finite fields is also true in F. We recall also that the property of being a pseudo-finite field is stable by ultraproducts.

5.4. **A brief review on quantifier elimination for Galois formulas.** Let A be an integral and normal scheme. A morphism of schemes $h : C \to A$ is a Galois cover if C is integral, h is étale (hence C is normal) and there is a finite group $G = G(C/A)$, the Galois group, acting on C such that A is isomorphic to the quotient C/G in such a way that h is the composition of the quotient morphism with the isomorphism. We say that the Galois cover $h : C \to A$ is colored if $G(C/A)$ is equipped with a family Con of subgroups of $G(C/A)$ which is stable by conjugation under elements of $G(C/A)$.
Let S be an integral normal scheme and let X_S be a variety over S. A normal stratification of X_S,

$$\mathcal{A} = \langle X_S, C_i/A_i \,|\, i \in I\rangle,$$

is a partition of X_S into a finite set of integral and normal locally closed S-subschemes A_i, $i \in I$, each equipped with a Galois cover $h_i : C_i \to A_i$.

A normal stratification $\mathcal{A}$ may be augmented to a Galois stratification

$$\mathcal{A} = \langle X_S, C_i/A_i, \mathrm{Con}(A_i) \,|\, i \in I \rangle,$$

if for each $i \in I$, $\mathrm{Con}(A_i)$ is a family of subgroups of $G(C_i/A_i)$ which is stable by conjugation under elements in $G(C_i/A_i)$, *i.e.* $(C_i \to A_i, \mathrm{Con}(A_i))$ is a colored Galois cover.

Let $U = \mathrm{Spec}\, R$ be an affine scheme, which we assume to be integral and normal. For any variety X_U over U and any closed point x of U, we denote by $\mathbf{F}_x$ the residual field of x on U and by X_x the fiber of X_U at x. More generally, for any field M containing $\mathbf{F}_x$, we shall denote by $X_{x,M}$ the fiber product of X_U and $\mathrm{Spec}\, M$ over x.

Let X_U be a variety over U. Let $\mathcal{A} = \langle X_U, C_i/A_i, \mathrm{Con}(A_i) \,|\, i \in I \rangle$ be a Galois stratification of X_U and let x be a closed point of U. Let M be a field containing $\mathbf{F}_x$ and let $\mathbf{a}$ be an M-rational point of X_x belonging to $A_{i,x}$. We denote by $\mathrm{Ar}(C_i/A_i, x, \mathbf{a})$ the conjugacy class of subgroups of $G(C_i/A_i)$ consisting of the decomposition subgroups at $\mathbf{a}$. We shall write

$$\mathrm{Ar}(\mathbf{a}) \subset \mathrm{Con}(\mathcal{A})$$

for

$$\mathrm{Ar}(C_i/A_i, x, \mathbf{a}) \subset \mathrm{Con}(A_i).$$

For x a closed point of U and M a field containing $\mathbf{F}_x$, we consider the subset

$$Z(\mathcal{A}, x, M) := \Big\{ \mathbf{a} \in X_U(M) \,\Big|\, \mathrm{Ar}(\mathbf{a}) \subset \mathrm{Con}(\mathcal{A}) \Big\}$$

of $X_U(M)$.

Let $\mathcal{A} = \langle \mathbf{A}_U^{m+n}, C_i/A_i, \mathrm{Con}(A_i) \,|\, i \in I \rangle$ be a Galois stratification of $\mathbf{A}_U^{m+n}$. Let $Q_1, \ldots, Q_m$ be quantifiers. We denote by ϑ or by $\vartheta(\mathbf{Y})$ the formal expression

$$(Q_1 X_1) \ldots (Q_m X_m) \, [\mathrm{Ar}(\mathbf{X}, \mathbf{Y}) \subset \mathrm{Con}(\mathcal{A})]$$

with $\mathbf{X} = (X_1, \ldots, X_m)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$. We call $\vartheta(\mathbf{Y})$ a *Galois formula* over R in the free variables $\mathbf{Y}$.

Now to a Galois formula ϑ, to a closed point x of U and to a field M containing $\mathbf{F}_x$, one associates the subset

$$Z(\vartheta, x, M) := \\ \Big\{ \mathbf{b} = (b_1, \ldots, b_n) \in M^n \,\Big|\, (Q_1 a_1) \ldots (Q_m a_m) \, [\mathrm{Ar}(\mathbf{a}, \mathbf{b}) \subset \mathrm{Con}(\mathcal{A})] \Big\}$$

of M^n, where the quantifiers $Q_1 a_1, \ldots, Q_m a_m$ run over M.

Let $\varphi(Y_1, \ldots, Y_n)$ be a formula in the first order language of rings with coefficients in the ring R and free variables $Y_1, \ldots, Y_n$. For every closed point x in U and every field M containing $\mathbf{F}_x$ we denote by $Z(\varphi, x, M)$ the subset of M^n defined by the (image over $\mathbf{F}_x$ of the) formula φ. Assume now φ is in prenex normal form, *i.e.* a formula of the form

$$(Q_1 X_1) \ldots (Q_m X_m) \Big[\bigvee_{i=1}^{k} \bigwedge_{j=1}^{l} f_{i,j}(\mathbf{X}, \mathbf{Y}) = 0 \wedge g_{i,j}(\mathbf{X}, \mathbf{Y}) \neq 0 \Big], \tag{5.4.1}$$

with $f_{i,j}$ and $g_{i,j}$ in $R[\mathbf{X},\mathbf{Y}]$. The formula between brackets defines an U-constructible subset W of $\mathbf{A}_U^{m+n}$ to which one associates a Galois stratification by taking any normal stratification with all strata contained either in W or in its complement, by taking the identity morphism as Galois cover on each stratum, and taking for $\mathrm{Con}(A_i)$ the family consisisting of the group with one element if $A_i \subset W$ and the empty family otherwise. In this way one obtains a Galois formula ϑ over R such that $Z(\vartheta, x, M) = Z(\varphi, x, M)$ for every closed point x in U and every field M containing $\mathbf{F}_x$.

There exists several versions of quantifier elimination for Galois formulas [24], [22], [23]. We shall use the following one which is a special case of Proposition 25.9 of [23].

5.4.1. **Theorem** (Fried-Jarden). *Let k be a field. Let $\mathcal{A}$ be a Galois stratification of $\mathbf{A}_k^{m+n}$ and let ϑ be a Galois formula*

$$(Q_1X_1)\dots(Q_mX_m)\,[\mathrm{Ar}(\mathbf{X},\mathbf{Y})\subset \mathrm{Con}(\mathcal{A})]$$

with respect to $\mathcal{A}$. There exists a Galois stratification $\mathcal{B}$ of $\mathbf{A}_k^n$ such that, for every pseudo-finite field F containing k,

$$Z(\vartheta, \mathrm{Spec}\, k, F) = Z(\mathcal{B}, \mathrm{Spec}\, k, F).$$

5.4.2. **Corollary.** *Let $\varphi(Y_1,\dots,Y_n)$ be a formula in the first order language of rings with coefficients in a field k and free variables $Y_1,\dots,Y_n$. There exists a Galois stratification $\mathcal{B}$ of $\mathbf{A}_k^n$ such that, for every pseudo-finite field F containing k,*

$$Z(\varphi, \mathrm{Spec}\, k, F) = Z(\mathcal{B}, \mathrm{Spec}\, k, F).$$

5.5. **Assigning virtual motives to formulas.** Let k be a field of characteristic zero. Let us consider the Grothendieck ring $K_0(\mathrm{PFF}_k)$ of the theory of pseudo-finite fields over k as defined in 3.4. It follows from 3.2.5 that we have a canonical morphism $\chi_c : K_0(\mathrm{Var}_k) \to K_0(\mathrm{CHMot}_k)$. We shall denote by $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$ the image of $K_0(\mathrm{Var}_k)$ in $K_0(\mathrm{CHMot}_k)$ under this morphism. Remark that the image of $\mathbf{L}$ in $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$ is not a zero divisor since it is invertible in $K_0(\mathrm{CHMot}_k)$. We shall now explain the construction of a canonical ring morphism

$$\chi_c : \mathrm{K}_0(\mathrm{PFF}_k) \longrightarrow \mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k)\otimes \mathbf{Q}$$

extending the one in 3.2.5.

5.5.1. **Theorem** (Denef-Loeser [17],[19]). *Let k be a field of characteristic zero. There exists a unique ring morphism*

$$\chi_c : \mathrm{K}_0(\mathrm{PFF}_k) \longrightarrow \mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k)\otimes \mathbf{Q}$$

satisfying the following two properties:

(i) *For any formula φ which is a conjunction of polynomial equations over k, the element $\chi_c([\varphi])$ equals the class in $\mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k)\otimes \mathbf{Q}$ of the variety defined by φ.*

(ii) *Let X be a normal affine irreducible variety over k, Y an unramified Galois cover of X, and C a cyclic subgroup of the Galois group G of Y over X. For such data we denote by $\varphi_{Y,X,C}$ a ring formula, whose interpretation in any field K containing k, is the set of K-rational points on X that lift to a geometric point on Y with decomposition group C (i.e. the set of points on X that lift to a K-rational point of Y/C, but not to any K-rational point of Y/C' with C' a proper subgroup of C). Then*

$$\chi_c([\varphi_{Y,X,C}]) = \frac{|C|}{|\mathrm{N}_G(C)|}\chi_c([\varphi_{Y,Y/C,C}]),$$

where $\mathrm{N}_G(C)$ is the normalizer of C in G.

Moreover, when k is a number field, for almost all finite places $\mathfrak{P}$, $N_{\mathfrak{P}}(\chi_c([\varphi]))$ is equal to the cardinality of $h_\varphi(k(\mathfrak{P}))$.

The above Theorem is a variant of results in §3.4 of [17]. A sketch of proof is given in [19].

Some ingredients in the proof. Uniqueness uses quantifier elimination for pseudo-finite fields from which it follows by Corollary 5.4.2 that $\mathrm{K}_0(\mathrm{PFF}_k)$ is generated as a group by classes of formulas of the form $\varphi_{Y,X,C}$. Thus by (ii) we only have to determine $\chi_c([\varphi_{Y,Y/C,C}])$, with C a cyclic group. But this follows directly from the following recursion formula:

$$|C|\,[Y/C] = \sum_{A \,\text{subgroup of}\, C} |A|\,\chi_c([\varphi_{Y,Y/A,A}]). \tag{5.5.1}$$

This recursion formula is a direct consequence of (i), (ii), and the fact that the formulas $\varphi_{Y,Y/C,A}$ yield a partition of Y/C. The proof of existence is based on work of del Baño Rollin and Navarro Aznar [6] who associate to any representation over $\mathbf{Q}$ of a finite group G acting freely on an affine variety Y over k, an element in the Grothendieck group of Chow motives over k. By linearity, we can hence associate to any $\mathbf{Q}$-central function α on G (i.e. a $\mathbf{Q}$-linear combination of characters of representations of G over $\mathbf{Q}$), an element $\chi_c(Y,\alpha)$ of that Grothendieck group tensored with $\mathbf{Q}$. Using Emil Artin's Theorem, that any $\mathbf{Q}$-central function α on G is a $\mathbf{Q}$-linear combination of characters induced by trivial representations of cyclic subgroups, one shows that $\chi_c(Y,\alpha) \in \mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k)\otimes\mathbf{Q}$. For $X := Y/G$ and C any cyclic subgroup of G, we define $\chi_c([\varphi_{Y,X,C}]) := \chi_c(Y,\theta)$, where θ sends $g \in G$ to 1 if the subgroup generated by g is conjugate to C, and else to 0. With some more work we prove that the above definition of $\chi_c([\varphi_{Y,X,C}])$ extends by additivity to a well-defined map $\chi_c : \mathrm{K}_0(\mathrm{PFF}_k) \longrightarrow \mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k)\otimes\mathbf{Q}$. □

Clearly $\chi_c(\varphi)$ depends only on h_φ and the construction easily extends by additivity to definable subassignments of h_X, for any variety X over k. So, to any such definable subassignment h, we may associate $\chi_c(h)$ in $\mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k)\otimes\mathbf{Q}$.

The invariants Eu and H being cohomological they factor through $\mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k)$. One can show, cf. [20], that for any definable subassignment h, $\mathrm{Eu}(\chi_c(h))$ belongs to $\mathbf{Z}$. Such an integrality result does not hold for H as shown by the following example: Let

n be a integer ≥ 1 and assume k contains all n-roots of unity. Consider the formula $\varphi_n : (\exists y)(x = y^n \text{ and } x \neq 0)$; then $\chi_c(\varphi_n) = \frac{\mathbf{L}-1}{n}$. In particular $\mathrm{Eu}(\chi_c(\varphi_n)) = 0$ and $H(\chi_c(\varphi_n)) = \frac{uv-1}{n}$. (This example contradicts the example on page 430 line -2 of [17] which is unfortunately incorrect.)

6. Arithmetic motivic integration

6.1. **The series P_{ar}.** We now consider the series

$$P_{\mathrm{ar}}(T) := \sum_{n \geq 0} \chi_c(\pi_n(h_{\mathcal{L}(X)}))\, T^n$$

in $\mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$.

6.1.1. **Theorem** (Denef-Loeser [17]). *Assume* $\mathrm{char} k = 0$.

1) *The series* $P_{\mathrm{ar}}(T)$ *in* $\mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q}$ *is rational of the form*
$$\frac{R(T)}{\prod(1 - \mathbf{L}^a T^b)},$$
with $R(T)$ *in* $(\mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbf{Q})[T]$, *a in* $\mathbf{Z}$ *and b in* $\mathbf{N} \setminus \{0\}$.
2) *If X is defined over some number field K, then, for almost all finite places* $\mathfrak{P}$,
$$N_{\mathfrak{P}}(P_{\mathrm{ar}}(T)) = P_{X \otimes \mathcal{O}_{K_{\mathfrak{P}}}}(T).$$

Here we use implicitely that $N_{\mathfrak{P}}$ factors through $\mathrm{K}_0^{\mathrm{mot}}(\mathrm{Var}_k)$ which follows from the fact there exists, by Grothendieck trace formula, a cohomological expression for $N_{\mathfrak{P}}$. The proof of Theorem 6.1.1 relies on the theory of arithmetic motivic integration we shall now explain.

6.2. **Definable subassignments of $h_{\mathcal{L}(X)}$.** We shall use now the notations and definitions in 4.5, assuming that $K = k((t))$, that $\kappa = k$, with k a field of characteristic zero, and that ord and ac have their classical meaning for formal power series.

Let R be a subring of k. By an $\mathcal{L}_{\mathrm{Pas}}$-formula with coefficients in R in the valued field sort and in the residue field sort, we mean a formula in the language obtained from $\mathcal{L}_{\mathrm{Pas}}$ by adding, for every element of R, a new symbol to denote it in the valued field sort and in the residue field sort. We shall consider $\mathcal{L}_{\mathrm{Pas}}$-formulas with coefficients in R in the valued field sort and in the residue field sort, free variables $x_1, \ldots, x_m$ running over the valued field sort and no free variables running over the residue field or the value sort. We shall call such formulas formulas on $R[[t]]^m$ for short

One may deduce the following statement of Ax/Ax-Kochen-Eršov type from the Pas Theorem.

6.2.1. **Proposition.** *Let R be a normal domain of finite type over* $\mathbf{Z}$ *with field of fractions k. Let σ be a sentence in the language* $\mathcal{L}_{\mathrm{Pas}}$ *with coefficients in R in the valued field sort and in the residue field sort. The following statements are equivalent:*

(1) *The sentence σ is true in $F((t))$ for every pseudo-finite field F containing k.*
(2) *There exists f in $R \setminus \{0\}$ such that, for every closed point x in $\operatorname{Spec} R_f$, the sentence σ is true in $\mathbf{F}_x((t))$.*

If, furthermore, k is a finite extension of $\mathbf{Q}$, the previous statements are also equivalent to the following:

(3) *There exists f in $R \setminus \{0\}$, multiple of the discriminant of $k/\mathbf{Q}$, such that, for every closed point x in $\operatorname{Spec} R_f$, the sentence σ is true in k_x,*

where k_x denotes the completion of k at x. Remark that, the extension $k/\mathbf{Q}$ being non ramified at x, the field k_x admits a canonical uniformizing parameter, hence also a canonical angular component map.

Let k be a field and let X be a variety over k. We consider the functor $h_{\mathcal{L}(X)} : K \mapsto X(K[[t]])$ from Field_k to the category of sets.
Let φ be a formula on $k[[t]]^m$. For every field K in Field_k, denote by $Z(\varphi, K[[t]])$ the subset of of all x in $K[[t]]^m = \mathbf{A}_k^m(K[[t]])$ for which $\varphi(x)$ is true in $K((t))$. This defines a subassignment $K \mapsto Z(\varphi, K[[t]])$ of the functor $h_{\mathcal{L}(\mathbf{A}_k^m)}$. We call such a subassignment a definable subassignment of $h_{\mathcal{L}(\mathbf{A}_k^m)}$.
By using affine coverings, one may also define definable subassignments of $h_{\mathcal{L}(X)}$, for X any variety over k.
We shall denote by $\mathrm{Def}_k(\mathcal{L}(X))$ the set of definable subassignments of $h_{\mathcal{L}(X)}$. Clearly $\mathrm{Def}_k(\mathcal{L}(X))$ is stable by finite intersection and finite union and by taking complements.
For n in $\mathbf{N}$, recall the canonical truncation morphism $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$. Hence if h is a subassignment of $h_{\mathcal{L}(X)}$ (resp. of $h_{\mathcal{L}_n(X)}$) we may consider $\pi_n(h) : K \mapsto \pi_n(h(K))$ (resp. $\pi_n^{-1}(h) : K \mapsto \pi_n^{-1}(h(K))$) which is a subassignment of $h_{\mathcal{L}_n(X)}$ (resp. of $h_{\mathcal{L}(X)}$).

6.2.2. **Proposition.** *Let h be a definable subassignment of $h_{\mathcal{L}(X)}$. Then, for every n in $\mathbf{N}$, $\pi_n(h)$ is a definable subassignment of $h_{\mathcal{L}_n(X)}$ and $\pi_n^{-1}\pi_n(h)$ is a definable subassignment of $h_{\mathcal{L}(X)}$.*

6.3. **Arithmetic motivic integration.** Now let us explain briefly how arithmetic motivic integration is constructed. We shall denote by $\mathcal{M}_k^{\mathrm{mot}}$ the image of $\mathcal{M}_k$ in $K_0(\mathrm{CHMot}_k)$ by the morphism χ_c. We endow $\mathcal{M}_k^{\mathrm{mot}}$ with the filtration $F^\bullet$, image by χ_c of the filtration $F^\bullet$ on $\mathcal{M}_k$ and we denote by $\widehat{\mathcal{M}}_k^{\mathrm{mot}}$ the completion of $\mathcal{M}_k^{\mathrm{mot}}$ with respect to that filtration.
Arithmetic motivic integration will assign to subassignments h of $h_{\mathcal{L}(X)}$ a measure $\nu(h)$ in $\widehat{\mathcal{M}}_k^{\mathrm{mot}} \otimes \mathbf{Q}$. The idea of the construction is very much the same as the one in 4.3, starting from stable cylinders. Since we are concerned only with definable subassignments, we shall not talk about measurable subassignments here. For a definable subassignment h of $\mathcal{L}(X)$ which is a stable cylinder (this is defined in a completely similar way than in the geometric case), the sequence $\chi_c(\pi_n(h))\mathbf{L}^{-(n+1)d}$ has a constant value $\tilde{\nu}(h)$ in $\mathcal{M}_k^{\mathrm{mot}} \otimes \mathbf{Q}$ for large n, with d the dimension of X.

6.3.1. **Theorem-Definition** (Denef-Loeser). There exists a unique mapping

$$\nu : \mathrm{Def}_k(\mathcal{L}(X)) \longrightarrow \widehat{\mathcal{M}}_k^{\mathrm{mot}} \otimes \mathbf{Q}$$

satisfying the following properties.

(1) If h is a stable cylinder which is a definable subassignment of $h_{\mathcal{L}(X)}$, then $\nu(h)$ is equal to the image of $\tilde{\nu}(h)$ in $\widehat{\mathcal{M}}_k^{\mathrm{mot}} \otimes \mathbf{Q}$.

(2) If h and h' are definable subassignments of $h_{\mathcal{L}(X)}$, then

$$\nu(h \cup h') = \nu(h) + \nu(h') - \nu(h \cap h').$$

(3) If $h(E) = h'(E)$ for every pseudo-finite field E containing k, then $\nu(h) = \nu(h')$.

(4) Let h be a definable subassignment of $h_{\mathcal{L}(X)}$. If there exists a subvariety S of X with $\dim S \leq d-1$ such that $h \subset h_{\mathcal{L}(S)}$, then $\nu(h) = 0$.

(5) Let h_n be a definable partition of a definable subassignment h with parameter $n \in \mathbf{N}$. Then the series $\sum_{n\in\mathbf{N}} \nu(h_n)$ is convergent in $\widehat{\mathcal{M}}_k^{\mathrm{mot}} \otimes \mathbf{Q}$ and

$$\nu(h) = \sum_{n\in\mathbf{N}} \nu(h_n).$$

(6) Let h and h' be definable subassignments of $h_{\mathcal{L}(X)}$. Assume $h \subset h'$. If $\nu(h')$ belongs to $F^e\widehat{\mathcal{M}}_k^{\mathrm{mot}} \otimes \mathbf{Q}$, then $\nu(h)$ also belongs to $F^e\widehat{\mathcal{M}}_k^{\mathrm{mot}} \otimes \mathbf{Q}$. (Here $F^{\bullet}\widehat{\mathcal{M}}_k^{\mathrm{mot}} \otimes \mathbf{Q}$ denotes the filtration induced by $F^{\bullet}$ on $\widehat{\mathcal{M}}_k^{\mathrm{mot}} \otimes \mathbf{Q}$.)

We call $\nu(h)$ the arithmetic motivic volume of h.

We have the following analogue of Theorem 4.4.2:

6.3.2. **Theorem.** *Let X be a variety over k of dimension d. Let h be a definable subassignment of $h_{\mathcal{L}(X)}$. Then*

$$\lim_{n\to\infty} \chi_c(\pi_n(h))\, \mathbf{L}^{-(n+1)d} = \nu(h)$$

in $\widehat{\mathcal{M}}_k^{\mathrm{mot}} \otimes \mathbf{Q}$.

Also, if $\alpha : h \to \mathbf{N}$ is a definable function on the definable subassignment h, meaning that, for every field K containing k, we have a function $\alpha(K) : h(K) \to \mathbf{N}$ and that the graph of these functions are definable in $\mathcal{L}(X) \times \mathbf{N}$, we may consider the integral $\int_h \mathbf{L}^{-\alpha} d\nu$. In particular, we have a direct analogue of the change of variables formula (Theorem 4.6.1) for arithmetic motivic integration, with a similar proof relying on Proposition 4.6.2.
We have also general rationality Theorems, for which we refer to § 7 of [17]. Using Proposition 6.2.1, one may also prove general specialization results of arithmetic integrals to p-adic ones (cf. § 8 of [17]). In particular Theorem 6.1.1 concerning P_{ar} may be obtained as a consequence of these rationality and specialization statements. Here is a typical example of such a statement:

6.3.3**. Theorem** (Denef-Loeser [17])**.** *Let K be a number field. Let φ be a first order formula in the language $\mathcal{L}_{\mathrm{Pas}}$ with coefficients in K and free variables $x_1, \ldots, x_n$. Let f be a polynomial in $K[x_1, \ldots, x_n]$. For $\mathfrak{P}$ a finite place of K, denote by $K_{\mathfrak{P}}$ the completion of K at $\mathfrak{P}$. For almost all $\mathfrak{P}$, applying the operator $N_{\mathfrak{P}}$ to the motivic integral $\int_h \mathbf{L}^{-s(\mathrm{ord} f)} d\nu$ gives the p-adic integral*

$$\int_{h_\varphi(K_{\mathfrak{P}})} |f|^s_{\mathfrak{P}} |dx|_{\mathfrak{P}}$$

Let us remark that this result is sufficient to get the specialization statement in Theorem 6.1.1 about P_{ar}. Indeed, one may as well assume f is a definable function in Theorem 6.3.3, since by a graph construction one may always replace f by a coordinate.

6.4. **"All natural p-adic integrals are motivic".** Theorem 6.3.3 is an illustration of the principle "All natural p-adic integrals are motivic". It applies in particular to integrals occuring in p-adic harmonic analysis, like orbital integrals. This has led recently Tom Hales [31] to propose that many of the basic objects in representation theory should be motivic in nature and to develop a beautiful conjectural program aiming to the determination of the virtual Chow motives that should control the behavior of orbital integrals and leading to a motivic fundamental lemma (see [26] and [32] for recent progress on these questions).

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