A History of Interactions between Logic and Number Theory

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1 Outline

Interactions between logic and number theory have almost always involved definability.

In an earlier period the relevant logical component was recursion theory (decidability and undecidability). For $\mathbb{Z}$ the central issue was Hilbert's 10th Problem, and the central result is that recursively enumerable relations on $\mathbb{Z}$ are existentially definable. The highpoint of definability theory in $\mathbb{Q}$ remains Julia Robinson's, that $\mathbb{Z}$ is Π_3-definable in $\mathbb{Q}$. Whether $\mathbb{Z}$ is existentially definable in $\mathbb{Q}$ is unknown (if it is, Hilbert's 10th Problem for $\mathbb{Q}$ is undecidable).

Recursion theory is thus very relevant for the logic of global fields and their rings of integers.

In contrast, model theory is much more relevant for the logic of local fields, and for those areas of number theory with a geometric aspect.

The locally compact completions of number fields have all undergone fruitful model-theoretic analyses. Thus Tarski (1930's) obtained the classical results on definitions in $\mathbb{C}$ and $\mathbb{R}$, while not till the 1960's did Ax-Kochen-Ersov obtain analogous results for p-adic fields (and for many Henselian fields). The completions are all decidable, but nowadays one gives more importance to the definability aspect of the above analyses. One obtains normal forms for definitions, links between the geometry of the set and the form of its definition, and various uniformities for number of connected components, and in $\mathbb{Q}_p$ for the form of various integrals (cf. Loeser's course).

In the 1980's the ring of all algebraic integers was shown (via a local-to-global principle involving earlier work in algebraically closed fields with valuation) to have a very clear definability theory, and in particular to have the analogue of Hilbert's 10th Problem decidable.

In the 1960's and 70's there were several developments in pure model theory that led some time later to interactions with number theory. The first was the work of Robinson school on model completion, existentially closed structures, and forcing methods. The theories of the completions $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}_p$ all arise naturally in this settings. But other theories emerge, without natural models, but which were to be key components in significant interactions in the 1990's. One is the theory of differentially closed fields (to be involved in the Mordell- Lang conjecture, cf. the Pillay-Scanlon course), and another (not discovered till 1990) is the theory ACFA of generic automorphisms (to be involved in the "logical" approach to the Manin- Mumford conjecture). Other theories of this type relate to the lifting of Frobenius to the Witt vectors.

The other model-theoretic development, certainly deeper qua model theory, originated with Morley's (1965) work initiating model theoretic stability. The 1965 paper gave a suggestive topological setting for first-order definability,and initiated a systematic study of general notions around the geometrical ideas of dimension and independence. Although the ensuing stability theory applies only to $\mathbb{C}$ among the completions of number fields, latter day "local" versions of it have been involved in most recent interactions of logic and number theory (cf. Pillay-Scanlon).

The other turning point in the 60's was Ax's work on the elementary theory of finite fields,where logic was seen to interact with Weil's Riemann Hypothesis for curves, and with Cebotarev's Theorem. The new theories could be construed as completions of Robinson type, and their definability was suggestively analyzed in terms of Galois Stratification (cf. Loeser's lectures). Moreover, one was soon led to model theoretic questions about absolute Galois groups, and those are related to a vision of Grothendieck (see Pop's lectures). (This is by no means the only case where ideas of Grothendieck, the "logician" of Bourbaki, have had fundamental logical content.)

Already in Ax-Kochen-Ersov one had seen the power of the impulse "Let p go to 0." After Ax's work one had richer environments for this idea. In particular, one could gradually approach the study of the Frobenius x^p as p tends to 0. Here one makes essential contact with the Weil Conjectures and the cohomological methods used in their proof. Old themes of Robinson, on bounds in the theory of ideals in polynomial rings, reappear in the wider setting of Intersection Theory and Weil Cohomology, and relate to the Grothendieck Standard Conjectures.

Another grand conjecture, that of Schanuel on transcendence of values of the complex exponential, has recently begun to interact with logic. It was first seen in connection with the decidability of the real and p-adic exponentials, and more recently in a profound definability-theoretic study by Zilber of the undecidable complex exponential. Zilber's work interacts naturally with diophantine geometry, and with old work of Ax (cf. Pillay-Scanlon).

2 Structure of lectures

1. Definability in the fields $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}_p$.

   Uniformities, with special reference on those in p.

2. Model theory. New theories. Ax's work.

3. From pseudofinite to ACFA. Galois groups and logic. Manifestations of Frobenius.

4. Analytic aspects. Schanuel's Conjecture.

3 Prerequisites

Basics of first-order logic, algebraic geometry, and number theory. (Roughly as for Poonen's course).

4 Project

The completions of number fields are naturally united in the adele construction. 25 years ago Weispfenning gave a first analysis of definability in the adeles, in the spirit of Feferman-Vaught Theorems. In view of the deepening of our understanding of definability in the interim (e.g., in the work of Denef and Loeser), it seems natural to go back and write an up-to-date account, paying attention to measure-theoretic and analyic uniformities. For example, what exactly do the model theoretic uniformities for p-adic integrals mean in an adelic setting? The goal is to give a Feferman-Vaught analysis for the analytic structure of the adeles.

I am currently having trouble tracking down the reference for Weispfenning's original work, but am confident of locating it before long.

References

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