COHOMOLOGY, PERIODS AND THE HODGE STRUCTURE OF TORIC HYPERSURFACES

VICTOR V. BATYREV

Let f be a Laurent polynomial considered as a regular function on a d-dimensional algebraic torus $\mathbb{T}^d$. The aim of these notes is to explain some ideas in the study of cohomology groups $H^*(Z_f)$ of nondegenerate affine toric hypersurfaces Z_f defined by the equation $f = 0$. The central role is played by the differential equations for periods of Z_f. We explain the relation of periods of Z_f to the GKZ-hypergeometric functions and discuss their applications to number theory and toric mirror symmetry.

Contents

1. Introduction

Let $M \cong \mathbb{Z}^d$ be a free abelian group of rank d, A a finite set in M, K is an arbitrary field and $\{a_m\}_{m\in A}$ a finite sequence of elements in K parametrized by A.

We consider a Laurent polynomial

$$f(x) = f(x_1, \ldots, x_d) = \sum_{m\in A} a_m x^m \in K[M],$$

as a regular function on the d-dimensional algebraic torus $\mathbb{T} = \mathrm{Spec}\,[M]$. The purpose of these lectures is to explain a common combinatorial framework related to f which appears in different areas: complex algebraic geometry, number theory and theoretical physics.

We denote by Δ the convex hull of A in $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

In section 2 we discuss some explicit Zariski open condition (so called Δ-nondegeneracy) on the coefficients $\{a_m\}_{m\in A}$ of f considered as points in $\mathbb{A}_K^{|A|}$. This condition is equivalent to nonvanishing of a polynomial $E_A(f)$ which is called **principal A-determinant of f**. The polynomial $E_A(f)$ was introduced by Gelfand, Kapranov and Zelevinski [25].

In section 3 we explain the main result in [25] about the Newton polytope of $E_A(f)$ which is called a **secondary polytope of** A.

The set $Z_f \subset \mathbb{T}^d$ of solutions of the equation $f = 0$ depends on the coefficients $\{a_m\}_{m\in A}$. In section 4 we consider the mixed Hodge structure in cohomology groups of complex toric hypersurfaces $Z_f \subset \mathbb{T}^d_{\mathbb{C}} \cong (\mathbb{C}^*)^d$ and explain some formulas for Hodge-Deligne numbers of Z_f which were obtained by Danilov and Khovanski [17]. Instead of zero set Z_f of a Laurent polynomial f one can consider the equations

$$f(x) - t = 0,$$

where t is sufficiently large complex number. In this way we obtain a 1-parameter family of affine hypersurfaces $Z_{f_t} \subset \mathbb{T}^d_{\mathbb{C}}$ which are fibers of a locally trivial fibration defined by $f : \mathbb{T}^d_{\mathbb{C}} \to \mathbb{C}$ over the set $U_r := \{t \in \mathbb{C} \ : \ |t| > r\}$ for a sufficiently large positive real number r. The cohomology $H^*(Z_{f_t}$ together with the natural action of the monodromy around $t = \infty$ can be described in terms of the lattice polytope

$$\Delta_\infty(f) := \mathrm{Conv}(\{0\} \cup A)$$

which is called **Newton polytope of f at infinity** [39].

In section 5 we consider A-hypergeometric systems introduced by Gelfand, Kapranov and Zelevinski in [26]. If all elements of A belong to an affine hyperplane $\langle *, u\rangle = 1$ for some element u of the dual lattice $N = M^*$, then the corresponding A-hypergeometric system has only regular singularities described by the principal A-determinant. A general case of A-hypergeometric system motivated expenential (oscillating) integrals has been investigated by Dwork-Loeser [22, 23] and Adolphson [1].

In section 6 we give a short review of the p-adic method of Dwork [38]. This method allows to compute zeta-functions of hypersurfaces Z_f over finite fields and L-functions of exponential sums using p-adic analysis [41]. Various generalizations of Dwork's method were obtained by Adolphson and Sperber [2, 3, 4, 5].

In section 7 we describe method of Viro for describing topological varieties $Z_f(\mathbb{R})$ obtained as real zero locus of Laurent polynomials $f \in \mathbb{R}[M]$ [47]. This method has an extension to the case of complex hypersurfaces $Z_f(\mathbb{R})$ [40].

In section 8 we will explain combinatorial ideas behind so called **toric mirror symmetry** [10].

Im section 9 we give a list of some interesting questions which could be investigated by students attending the lectures.

In these lectures we will avoid all technical proofs. Sometimes we restrict ourselves to only explanation of main ideas and give some illustrating examples.

2. Principal A-determinant

Let $M \cong \mathbb{Z}^d$ be a free abelian group of rank d and $\mathbb{T}^d := \operatorname{Spec} K[M]$ the d-dimensional algebraic torus with the lattice of algebraic characters M, where K is an arbitrary field. Let A be a finite set in M and $\{a_m\}_{m\in A}$ a finite sequence of elements in K parametrized by A. Consider the Laurent polynomial

$$f(x) = f(x_1, \ldots, x_d) = \sum_{m\in A} a_m x^m \in K[M].$$

Our basic combinatorial object is the Newton polytope $\Delta = \Delta(f)$ of f:

Definition 2.1. The **Newton polytope** $\Delta(f)$ of f is defined to be the convex hull of all $m \in M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$ such that $a_m \neq 0$. If θ is a face of $\Delta(f)$ we define the Laurent polynomial

$$f_\theta := \sum_{m\in A\cap\theta} a_m x^m.$$

Definition 2.2. A Laurent polynomial $f(x)$ is said to be **nondegenerate with respect to its Newton polytope** $\Delta = \Delta(f)$ (or **Δ-nondegenerate**) if for any face $\theta \subset \Delta$ the system of the polynomial equations

$$f_\theta(x) = x_1 \frac{\partial}{\partial x_1} f_\theta(x) = \cdots = x_d \frac{\partial}{\partial x_d} f_\theta(x) = 0$$

has no solutions in $\mathbb{T}^d$ for any extension of K.

It is easy to see that Δ-nondegenerate Laurent polynomials $f(x) = f(x,a)$ form a Zariski dense open subset in the affine space $\mathbb{A}^{|A|}$ parametrizing the coefficients $\{a_m\}_{m\in A}$.

The Δ-nondegeneracy condtion for f can be expressed in geometric terms using a natural toric compactification $\mathbb{P}_\Delta$ of $\mathbb{T}$ [24].

Let $A = \{v_1, \ldots, v_n\}$. If the vectors $v_2 - v_1, v_3 - v_1, \ldots, v_n - v_1$ generate the lattice M, then the projective toric variety $\mathbb{P}_\Delta$ can be defined as the normalization of the projective closure of $\mathbb{T}$ under the embedding

$$\begin{aligned} \mathbb{T}^d &\hookrightarrow \mathbb{P}^{|A|-1} \\ x &\mapsto (x^{v_1} : x^{v_2} : \cdots : x^{v_n}). \end{aligned}$$

In general situation, $\mathbb{P}_\Delta$ is defined as a projective variety

$$\mathbb{P}_\Delta := \operatorname{Proj} S_\Delta$$

corresponding to the graded commutative K-algebra

$$S_\Delta = \bigoplus_{k\geq 0} S_\Delta^k \subset K[\mathbb{Z} \oplus M] \cong K[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$$

associated with the monoid of integral points in the cone $C_\Delta \subset \mathbb{R} \oplus M_{\mathbb{R}}$, where

$$C_\Delta := \sum_{i=1}^n \mathbb{R}_{\geq 0}(1, v_i).$$

In particular, S_Δ^k is a finite dimensional K-vector space with the basis $x_0^k x^m$ ($m \in k\Delta \cap M$).

The torus orbits $\mathbb{T}_\theta$ in $\mathbb{P}_\Delta$ are in 1-to-1-correspondence to faces $\theta \subset \Delta$. Denote by $\overline{Z}_f$ the closure of $Z_f \subset \mathbb{T}$ in $\mathbb{P}_\Delta$. A Laurent polynomial f is Δ-nondegenerate if and only if $Z_{\theta,f} := \mathbb{T}_\theta \cap \overline{Z}_f$ is a smooth hypersurface of codimension one in $\mathbb{T}_\theta$ for all faces $\theta \subset \Delta$.

Example 2.3. Let $d = 1$, $M = \mathbb{Z}$, $A = \{0, 1, \dots, n\}$ and $\Delta = [0, n]$. A polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

is Δ-nondegenerate if and only if $a_0 a_n \neq 0$ and f has exactly n distinct nonzero roots $\rho_1, \dots, \rho_n$ in the algebraic closure of K.

Example 2.4. Let $d = 2$, $M = \mathbb{Z}^2$, $\Delta = \mathrm{Conv}((0,0),(n,0),(0,n))$, $A = \Delta \cap M$. A polynomial

$$f(x_1, x_2) = \sum_{(m_1,m_2)\in A} \alpha_{m_1,m_2} x_1^{m_1} x_2^{m_2}$$

is Δ- nondegenerate if and only if the projective closure $\overline{Z}_f$ of Z_f in $\mathbb{P}^2$ is a smooth projective algebraic curve of degree n which does not contain 3 points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ and has transversal intersections with the 3 coordinate lines in $\mathbb{P}^2$ defined by the equations $z_i = 0$ $(i = 0,1,2)$. We remark that in this case the toric variety $\mathbb{P}_\Delta \hookrightarrow \mathbb{P}^{|A|-1}$ can be identified with n-th Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^{|A|-1}$.

For us it will be very important the following equivalent characterization of Δ-nondegenerate Laurent polynomials.

Theorem 2.5. *A Laurent polynomial f is Δ-nondegenerate if and only if $d+1$ polynomials*

$$F_0 = x_0 f(x), \quad F_i := x_i \frac{\partial}{\partial x_i} x_0 f(x), \quad 1 \leq i \leq d$$

form a regular sequence in S_Δ, i. e. the multiplication by F_i induces an injective endomorphism of the S_Δ-module

$$S_\Delta / \langle F_0, F_1, \dots, F_{i-1} \rangle S_\Delta$$

for all i $(1 \leq i \leq d)$.

The proof of this theorem uses the fact that S_Δ is a Cohen-Macaulay ring [35]. There is a standard reformulated of 2.5 using some homological algebra.

We denote by N the dual to M lattice $\mathrm{Hom}(M, \mathbb{Z})$. Then $\widetilde{N} := \mathbb{Z} \oplus N$ can de identified with the dual to lattice $\widetilde{M} := \mathbb{Z} \oplus M$. We denote by

$$\langle *, * \rangle \ : \ \widetilde{M} \times \widetilde{N} \to \mathbb{Z}$$

the canonical pairing. Every element $u \in \widetilde{N}$ defines an element ∂_u in the Lie algebra of the $(d+1)$-dimensional torus $\widetilde{\mathbb{T}} := \mathrm{Spec}\, K[\widetilde{M}]$ which acts on monomials x^μ $(\mu \in \widetilde{M})$ as follows

$$\partial_u (x^\mu) = \langle \mu, u \rangle x^\mu.$$

Consider the Koszul complex $S_\Delta(-*) \otimes_{\mathbb{Z}} \Lambda^* \widetilde{N}$ of graded S_Δ-modules

$$0 \to S_\Delta(-d-1) \otimes_{\mathbb{Z}} \Lambda^{d+1} \widetilde{N} \to S_\Delta(-d) \otimes_{\mathbb{Z}} \Lambda^d \widetilde{N} \to \cdots \to S_\Delta(-r) \otimes_{\mathbb{Z}} \Lambda^r \widetilde{N} \to$$
$$\to S_\Delta(-r+1) \otimes_{\mathbb{Z}} \Lambda^{r-1} \widetilde{N} \to \cdots \to S_\Delta(-1) \otimes_{\mathbb{Z}} \Lambda^1 \widetilde{N} \to S_\Delta \otimes_{\mathbb{Z}} \Lambda^0 \widetilde{N}.$$

The differential $d \ : \ S_\Delta(-r) \otimes_{\mathbb{Z}} \Lambda^r \widetilde{N} \to S_\Delta(-r+1) \otimes_{\mathbb{Z}} \Lambda^{r-1} \widetilde{N}$ is defined on generators $u_{i_0} \wedge u_{i_1} \wedge \cdots \wedge u_{i_r}$ as follows

$$d\,(u_{i_0} \wedge u_{i_1} \wedge \cdots \wedge u_{i_r}) = \sum_{s=0}^{r} (-1)^s \partial_{u_s} F \wedge u_{i_1} \wedge \cdots \wedge \widehat{u_{i_s}} \wedge \cdots \wedge u_{i_r},$$

where $F := x_0 f(x) - 1$.

Theorem 2.6. *A Laurent polynomial f is Δ-nondegenerate if and only if the Koszul complex $S_\Delta(-*) \otimes_{\mathbb{Z}} \Lambda^* \widetilde{N}$ is a free resolution of the Artinian S_Δ-module*

$$S_f := S_\Delta / \langle F_0, F_1, \ldots, F_d \rangle S_\Delta.$$

Since the graded Artinian S_Δ-module S_f may have only finitely nonzero homogeneous components, one can show that the last theorem can be reformulated in the following form:

Theorem 2.7. *A Laurent polynomial f is Δ-nondegenerate if and only if the l-th homogeneous component of $S_\Delta(-*) \otimes_{\mathbb{Z}} \Lambda^* \widetilde{N}$ is an exact complex of K-vector spaces*

$$C^{(k)}_{\cdot}(f) \; : \; 0 \to S_\Delta^{k-d-1} \otimes_{\mathbb{Z}} \Lambda^{d+1} \widetilde{N} \to S_\Delta^{k-d} \otimes_{\mathbb{Z}} \Lambda^{d} \widetilde{N} \to \cdots \to S_\Delta^{k-r} \otimes_{\mathbb{Z}} \Lambda^{r} \widetilde{N} \to$$
$$\to S_\Delta^{k-r+1} \otimes_{\mathbb{Z}} \Lambda^{r-1} \widetilde{N} \to \cdots \to S_\Delta^{k-1} \otimes_{\mathbb{Z}} \Lambda^{1} \widetilde{N} \to S_\Delta^{k} \otimes_{\mathbb{Z}} \Lambda^{0} \widetilde{N} \to 0$$

for all sufficiently large l.

Now we need the notion of the determinant of a complex.

Definition 2.8. Let R be an integral domain and K the field of fractions of R. Consider a finite complex $C.$ of locally free R modules

$$0 \to C_k \to C_{k-1} \to \cdots \to C_1 \to C_0 \to 0$$

such that the complex of K-vector spaces $C. \otimes_R K$ is exact. Then there exists a locally free R-submodule $\det C.$ of rank 1 in K (i.e., $\det C. \subset K$ is a fractional ideal in K) which is called the **determinant of the complex** $C.$.

In order to define $\det C. \subset K$ we can assume without loss of generality that all R-modules C_i $(0 \leq i \leq k)$ are free. Then we can use the induction on k and define $\det C.$ as follows.

Let $e_1, \ldots, e_s$ be a R-basis of C_1. If the images of $e_1, \ldots, e_r$ $(r \leq s)$ in C_0 form a K-basis of $C_0 \otimes_R K$, then we can split C_1 onto a direct sum

$$C_1 = C_1' \oplus C_1'', \;\; C_1' := \bigoplus_{i=1}^{r} Re_i, \;\; C_1'' := \bigoplus_{i=r+1}^{s} Re_i.$$

Then $\det C.$ is defined as

$$\det C. := \det\{C_1' \to C_0\} \cdot (\det C'_{\cdot})^{-1},$$

where $\det\{C_1' \to C_0\}$ is the usual determinant of the matrix of the R-module homomorphism $C_1' \to C_0$ and $C'_{\cdot}$ a shorter complex of free R-modules

$$0 \to C_k \to C_{k-1} \to \cdots \to C_2 \to C_1'' \to 0$$

obtained from $C.$ using the canonical projection $C_1 \to C_1''$.

One can show that R-submodule $\det C. \subset K$ does not depend on a choice of the above splitting $C_1 = C_1' \oplus C_1''$.

We apply above definition to the polynomial ring $R := \mathbb{Z}[a_1, \ldots, a_n]$, where $a_1, \ldots, a_n$ are considered as independent variables (i.e., algebraically independent elements over $\mathbb{Q}$). Let us set K to be the field of fractions of R and denote by $C^{(k)}_{\cdot}(f, R)$ the complex of free R-modules having the same ranks and the same differentials in the complex of K-vector spaces $C^{(k)}_{\cdot}(f)$ (see 2.7), i.e.,

$$C^{(k)}_{\cdot}(f, R) \otimes_R K \cong C^{(k)}_{\cdot}(f).$$

The following statement was proved by Gelfand, Kapranov and Zelevinsky in [26] (see also [28]).

Theorem 2.9. *For all sufficiently large l, the determinant of the complex $C^{(k)}_{\cdot}(f,R)$ is an ideal in R which does not depend on k.*

Remark 2.10. One can show that the theorem hold already for all $k \geq d+1$ (see 2.22).

Definition 2.11. The generator of the ideal $\det C^{(k)}_{\cdot}(f,R)$ for $k \geq d+1$ is called **principal** A**-determinant of** f and will be denoted by $E_A(f)$. We remark that $E_A(f)$ is uniquely determined up to multiplication by ± 1.

Using principal A-determinants, one obtains the following explicit characterization of Δ-nondegeneracy:

Theorem 2.12. *A Laurent polynomial f is Δ-nondegenerate if and only if $E_A(f) \neq 0$.*

Let us consider simplest examples.

Example 2.13. Let $d = 1$, $A = \{0,1,2\}$, $\Delta = [0,2]$. Then $\dim S^k_\Delta = 2k+1$ ($k \geq 0$) and the complex $C^{(2)}_{\cdot}(f)$ have the form

$$0 \to S^0_\Delta u_0 \wedge u_1 \xrightarrow{d_1} S^1_\Delta u_0 \oplus S^1_\Delta u_1 \xrightarrow{d_0} S^2_\Delta \to 0,$$

where the differentials d_0, d_1 are defined by the matrices D_0, D_1:

$$D_0 := \begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & a_1 & 0 & 0 \\ a_2 & a_1 & a_0 & 2a_2 & a_1 & 0 \\ 0 & a_2 & a_1 & 0 & 2a_2 & a_1 \\ 0 & 0 & a_2 & 0 & 0 & 2a_2 \end{pmatrix}$$

$$D_1 := \begin{pmatrix} 0 & -a_1 & -2a_2 & a_0, a_1, a_2 \end{pmatrix}.$$

Then the principal A-determinant $E_A(f)$ can be computed as ratio of two minors in D_0 and D_1 obtained by deleting 3-rd column from D_0 and choosing 3-rd element in D_1:

$$E_A(f) = \begin{vmatrix} a_0 & 0 & 0 & 0 & 0 \\ a_1 & a_0 & a_1 & 0 & 0 \\ a_2 & a_1 & 2a_2 & a_1 & 0 \\ 0 & a_2 & 0 & 2a_2 & a_1 \\ 0 & 0 & 0 & 0 & 2a_2 \end{vmatrix} (-2a_2)^{-1} = -a_0 \begin{vmatrix} a_0 & a_1 & 0 \\ a_1 & 2a_2 & a_1 \\ a_2 & 0 & 2a_2 \end{vmatrix} = a_0a_2(a_1^2 - 4a_0a_2)$$

Example 2.14. Let $d = 1$, $A = \{0,1,\ldots,n\}$, $\Delta = [0,n]$. Consider the complex $C^{(2)}_{\cdot}(f)$

$$0 \to S^0_\Delta u_0 \wedge u_1 \xrightarrow{d_1} S^1_\Delta u_0 \oplus S^1_\Delta u_1 \xrightarrow{d_0} S^2_\Delta \to 0$$

and use $\dim S^k_\Delta = nk+1$. Similar computation as above show that

$$E_A(f) = a_0 a_n D(f),$$

where $D(f)$ is the classical discriminant of f:

$$(1) \qquad D(f) := (-1)^{n(n-1)/2} \frac{1}{a_n} \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n & 0 & \cdots \\ 0 & a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} & a_n & \cdots \\ & \cdots & & \cdots & & \cdots & & \\ a_1 & 2a_1 & \cdots & (n-1)a_{n-1} & na_n & 0 & 0 & \cdots \\ 0 & a_1 & \cdots & (n-2)a_{n-2} & (n-1)a_{n-1} & na_n & 0 & \cdots \\ & \cdots & & \cdots & & \cdots & & \end{vmatrix}$$

In particular $E_A(f)$ is a homogeneous polynomial of degree $2n$.

The computation of $E_A(f)$ for $d > 1$ may be rather complicated. However we are able to give an explicit formula for $E_A(f)$ for some special subsets $A \subset M$.

Example 2.15. Let $A \subset \mathbb{Z}^d$ be the set of vectors $v_0, v_1, \ldots, v_d$, where $v_0 = 0$ and $v_1, \ldots, v_d$ is the standard basis of $\mathbb{Z}^d$. Then

$$f(x) = a_0 + a_1x_1 + \cdots + a_nx_d$$

is a linear function and $F_0 = a_0x_0$, $F_i = a_ix_0x_i$ $(1 \leq i \leq d)$. Since the Artinian ring S_f has nonzero elements only in degree 0, we can use the complex $C^{(1)}_{\cdot}(f)$ for the computation of $E_A(f)$ and obtain

$$E_A(f) = \pm a_0a_1 \cdots a_d.$$

It is not difficult to understand what happens with $E_A(f)$ if we replace M by some larger lattice M'.

Proposition 2.16. *Let us consider an arbitrary sublattice M' in $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ containing M. Denote by E'_A the principal A-determinant of f with respect to the new lattice M'. Then*

$$E'_A(f) = ([M' : M]E_A(f))^{[M':M]}.$$

Proof. The Koszul complex $C'_{\cdot}$ corresponding to the lattice M' splits into direct sum of $[M' : M]$ Koszul complexes corresponding to all representatives of M' modulo M, because S'_Δ is a free S_Δ-module of rank $[M' : M]$. Moreover, if $N' \subset N$ is the dual to M' sublattice, then $\Lambda^1(\mathbb{Z} \oplus N')$ is a $\mathbb{Z}$-submodule of finite index $[M' : M]$ in $\Lambda^1(\mathbb{Z} \oplus N)$. Combining both facts one obtains the demanded formula (we omit further details). $\square$.

Corollary 2.17. *If $A = \{v_0, \ldots, v_d\} \subset M$ is the set of vertices of a d-dimensional simplex Δ in $M_{\mathbb{R}}$, then*

$$E_A(f) = \pm \mathrm{Vol}(\Delta)^{\mathrm{Vol}(\Delta)}(a_0a_1 \cdots a_d)^{\mathrm{Vol}(\Delta)},$$

where $\mathrm{Vol}(\Delta)$ is $d!$ times the d-dimensional volume of Δ with respect to the lattice M.

Example 2.18. Let $A := \{(0, 0), (1, 0), (0, 1), (1, 1)\} \subset \mathbb{Z}^2$ and

$$f(x) := a_0 + a_1x_1 + a_2x_2 + a_3x_1x_2.$$

Then $\Delta = \mathrm{Conv}\, A$ is a simplest 2-dimensional polytope which is not a simplex. The computation of $E_A(f)$ shows that

$$E_A(f) = \pm a_0a_1a_2a_3(a_0a_3 - a_1a_2).$$

In order to understand better the homogeneous components of the Koszul complex $C_{\cdot}(f)$ which were used in the definition of $E_A(f)$ we will need a formula for the number $l(k\Delta)$ of lattice points in $k\Delta$ ($l(k\Delta)$ is the dimension of homogeneous component S^k_Δ of S_Δ).

Definition 2.19. For an arbitrary convex polytope $\Delta \subset M_{\mathbb{R}}$ with vertices in M we define two power series

$$P_\Delta(t) = \sum_{k \geq 0} l(k\Delta)t^k$$

and

$$Q_\Delta(t) = \sum_{k > 0} l^*(k\Delta)t^k,$$

where $l^*(k\Delta)$ denotes the number of lattice points in the interior of $k\Delta$.

Example 2.20. Assume that Δ is a d-dimensional simplex and $v_1, \ldots, v_{d+1}$ are its vertices. We denote by $\widetilde{M}'$ the sublattice in $\widetilde{M} = \mathbb{Z} \oplus M$ generated by $(1, v_1), \ldots, (1, v_{d+1})$. Then $r := [\widetilde{M} : \widetilde{M}'] = \mathrm{Vol}(\Delta)$. We can choose r canonical representatives $\mu_1, \ldots, \mu_r \in \widetilde{M}$ for the elements in the finite ablelian group $\widetilde{M}/\widetilde{M}'$ in the form

$$\mu_i = \lambda_1(1, v_1) + \cdots + \lambda_{d+1}(1, v_{d+1})$$

where $\lambda_1, \ldots, \lambda_{d+1}$ are non-negative rational numbers less than 1. The first coordinate of μ_i is a non-negative integer

$$w_i = \lambda_1 + \cdots + \lambda_{d+1} < d + 1.$$

Since $C_\Delta \cap \widetilde{M}$ is a disjoint union of r sets

$$\mu_i + \left(C_\Delta \cap \widetilde{M}'\right), \;\; i = 1, \ldots, r,$$

we obtain

$$P_\Delta(t) = \sum_{i=1}^{r} \frac{t^{w_i}}{(1-t)^{d+1}}.$$

Moreover, if C_Δ^* is the interior of the cone C_Δ, then $C_\Delta^* \cap \widetilde{M}$ is a disjoint union of r sets

$$\mu_i^* + C_\Delta \cap \widetilde{M}', \;\; i = 1, \ldots, r,$$

where

$$\mu_i^* := (d+1, v_1 + \cdots + v_{d+1}) - \mu_i, \;\; i = 1, \ldots, r,$$

Thus we have

$$Q_\Delta(t) = \sum_{i=1}^{r} \frac{t^{w_i^*}}{(1-t)^{d+1}},$$

where $w_i^* := d + 1 - w_i$ $(i = 1, \ldots, r)$. In particular,

$$\Psi_\Delta(t) := (1-t)^{d+1} P_\Delta(t) = \sum_{i \geq 0} \psi_i(\Delta) t^i = \sum_{i=1}^{r} t^{w_i}$$

and

$$\Phi_\Delta(t) := (1-t)^{d+1} Q_\Delta(t) = \sum_{i \geq 0} \varphi_i(\Delta) t^i = \sum_{i=1}^{r} t^{w_i^*}$$

are polynomials with non-negative integral coefficients satisfying the condition

$$t^{d+1}\Psi_\Delta(t^{-1}) = \Phi_\Delta(t),$$

i.e., $\varphi_i(\Delta) = \psi_{d+1-i}(\Delta)$ $(0 \leq i \leq d+1)$. Moreover, we have

$$r = \Psi_\Delta(1) = \sum_i \psi_i(\Delta) = \mathrm{Vol}(\Delta).$$

This example explicitly illustrate the Ehrhart reciprocity law which holds for arbitrary lattice polytopes Δ [16]:

Theorem 2.21. *Let Δ an arbitrary d-dimensional convex polytope $\Delta \subset M_{\mathbb{R}}$ with vertices in M. Then $P_\Delta(t)$ and $Q_\Delta(t)$ are rational functions such that*

$$\Psi_\Delta(t) := (1-t)^{d+1} P_\Delta(t)$$

and

$$\Phi_\Delta(t) := (1-t)^{d+1} Q_\Delta(t)$$

are polynomials of with nonnegative coefficients satisfying the condition

$$t^{d+1}\Psi_\Delta(t^{-1}) = \Phi_\Delta(t),$$

or equivalently

$$Q_\Delta(t) = (-1)^{d+1} P_\Delta(t^{-1}).$$

The polynomial $\Psi_\Delta(t)$ has degree $\leq d$ and

$$\Psi_\Delta(1) = \sum_i \psi_i(\Delta) = \sum_i \varphi_i(\Delta) = \Phi_\Delta(1) = \mathrm{Vol}(\Delta).$$

Proof. Consider a triangulation $\mathcal{T}$ of Δ into simplices $\{\tau_i\}$ with vertices in A. Some of such triangulations will be important in the sequel (see 3.13). Using inclusion-exclusion formula, we reduce the counting of lattice points in $k\Delta$ and in the interior of $k\Delta$ to the case of simplices (see 2.20).

$$Q_\Delta(t) = \sum_{\tau \in \mathcal{T}} Q_\tau(t) = \sum_{\tau \in \mathcal{T}} (-1)^{\dim \tau + 1} P_\tau(t^{-1}) =$$

$$(-1)^{d+1}\left(\sum_{\tau\in\mathcal{T}} (-1)^{d-\dim\tau} P_\tau(t^{-1})\right) = (-1)^{d+1} P_\Delta(t^{-1}),$$

where the sum runs over all simplices in $\mathcal{T}$ which are not contained in the boundary $\partial\Delta$ of Δ. Since for any simplex $\tau \in \mathcal{T}$ the polynomial $(1-t)^{\dim\tau+1}P_\tau(t)$ has degree $\leq \dim\tau$, we obtain that

$$\Psi_\Delta(t) = (1-t)^{d+1}P_\Delta(t) = (1-t)^{d+1}\left(\sum_{\sigma\in\mathcal{T}} (-1)^{d-\dim\tau} P_\tau(t)\right) = \sum_{\tau\in\mathcal{T}} (t-1)^{d-\dim\tau}\Psi_\tau(t)$$

is a polynomial of degree $\leq d$. The nonnegativity of the coefficients $\psi_i(\Delta)$ of $\Psi_\Delta(t)$ follows from the fact that $\psi_i(\Delta)$ is the dimension of the homogeneous component S_f^i of the Artinian ring S_f. Moreover, we have

$$\Psi_\Delta(1) = \sum_{\tau\in\mathcal{T}(d)} \Psi_\tau(1)$$

where the sum runs over the set $\mathcal{T}(d)$ all d-dimensional simplices in $\mathcal{T}$. Therefore,

$$\Psi_\Delta(1) = \sum_{\tau\in\mathcal{T}(d)} \mathrm{Vol}(\tau) = \mathrm{Vol}(\Delta).$$

□

Corollary 2.22. *The homogeneous components S_f^i of the Artinian ring S_f are zero for $i \geq d+1$.*

Remark 2.23. The Ehrhart reciprocity law can be interpreted from view-point of commutative algebra. Let us denote by I_Δ the ideal in S_Δ generated as K-vector space by monomials $x_0^k x^m \in S_\Delta$ such that m is an interior lattice point of $k\Delta$. Then $Q_\Delta(t)$ is a generating function for dimensions of homogeneous components I_Δ^k of I_Δ. It turns out that I_Δ is the **dualising module** for S_Δ. In particular, a regular sequence $F_0, F_1, \ldots, F_d$ in S_Δ will be also regular for I_Δ, i.e., the multiplication by F_i induces an injective endomorphism of the S_Δ-module

$$I_\Delta / \langle F_0, F_1, \ldots, F_{i-1} \rangle I_\Delta$$

for all i ($1 \leq i \leq d$). The Artinian S_f-module

$$I_f := I_\Delta / \langle F_0, F_1, \ldots, F_d \rangle I_\Delta$$

is a finite dimensional K-vector space. The generating function for dimensions of homogeneous components of I_f is exactly the polynomial $\Phi_\Delta(t)$ from the above theorem. The multiplication $S_f \times I_f \to I_f$ induces a perfect pairing between the homogeneous components of S_f and I_f:

$$S_f^i \times I_f^{d+1-i} \to I_f^{d+1} \cong K,$$

where the canonical isomorphism $I_f^{d+1} \cong K$ is defined by **toric residue** [14]. The isomorphism

$$S_f^i \cong \left(I_f^{d+1-i} \right)^*, \ \ 0 \leq i \leq d,$$

implies already known equations

$$\psi_i(\Delta) = \varphi_{d+1-i}(\Delta), \ \ 0 \leq i \leq d.$$

Here is one more general statement about $E_A(f)$:

Theorem 2.24. *Let f be a Laurent polynomial with the Newton polytope Δ. The principal A-determinant $E_A(f)$ is a homogeneous polynomial of the coefficients $\{a_m\}_{m \in A}$ of degree*

$$(d+1)\mathrm{Vol}(\Delta).$$

Proof. We remark that the differentials in the complexes $C^{(k)}l.(f)$ are defined by matrices whose coefficients are homogeneous polynomials of degree 1. If $r_i^{(k)}$ denotes the rank of $C_i^{(k)}(f)$ $(0 \leq i \leq d+1)$, then it follows from the computation of det $C_\cdot^{(k)}(f)$ as alternating product of minors that the degree of det $C_\cdot^{(k)}(f)$ for $k \geq d+1$ equals

$$g_k = r_0^{(k)} - (r_1^{(k)} - r_0^{(k)}) + (r_2^{(k)} - r_1^{(k)} + r_0^{(k)}) - (r_3^{(k)} - r_2^{(k)} + r_1^{(k)} - r_0^{(k)}) + \cdots$$
$$\cdots + (-1)^d (r_d^{(k)} - r_{d-1}^{(k)} + \cdots + (-1)^d r_d^{(k)}) = (d+1) r_0^{(k)} - d r_1^{(k)} + (d-1) r_2^{(k)} - \cdots + (-1)^d r_d^{(l)}.$$

If we set $G_\Delta(t) := \sum_{i \geq 0} g_i t^i$, then we obtain

$$G_\Delta(t) = (d+1)(1-t)^d P_\Delta(t) = \frac{(d+1)\Psi_\Delta(t)}{1-t},$$

since

$$r_i^{(k)} = \binom{d+1}{i} l((k-i)\Delta)$$

and

$$\sum_{i=0}^{d} (-1)^i (d+1-i) \binom{d+1}{i} t^{d-i} = ((t-1)^{d+1})' = (d+1)(t-1)^d,$$

i.e.,

$$\sum_{i=0}^{d}(-1)^i(d+1-i)\binom{d+1}{i}t^i = t^d(d+1)(t^{-1}-1)^d = (d+1)(1-t)^d.$$

Therefore,

$$g_k = (d+1)\Psi_\Delta(1) = (d+1)\mathrm{Vol}(\Delta) \ \ \forall k \geq d+1.$$

□

3. The secondary polytope

We start this section with the classical topic.

Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ be a general polynomial of degree n and $\rho_1, \dots, \rho_n$ its roots. Recall that the classical discriminant $D(f)$ of f can be also defined by the formula

$$D(f) := a_n^{2n-2}\prod_{i<j}(\rho_i - \rho_j)^2.$$

Using the main theorem about symmetric functions, one can write $D(f)$ as polynomial function $D(a_0, \dots, a_n)$ of $a_0, \dots, a_n$.

Example 3.1. For $n = 2$, one has

$$D(a_0+a_1x+a_2x^2) = a_2^2(\rho_1-\rho_2)^2 = a_2^2[(\rho_1+\rho_2)^2-4\rho_1\rho_2] = a_2^2\left[\left(\frac{a_1}{a_2}\right)^2 - 4\frac{a_1}{a_2}\right] = a_1^2-4a_0a_2.$$

With a little bit more work one obtains for $n = 3$

$$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_3^4(\rho_1 - \rho_2)^2(\rho_2 - \rho_3)^2(\rho_1 - \rho_3)^2 =$$
$$= a_1^2a_2^2 - 4a_1^3a_3 - 4a_0a_2^3 - 27a_0^2a_3^2 + 18a_0a_1a_2a_3.$$

The case $n = 4$ is much more complicated for the computation, because the polynomial $D(a_0, \dots, a_4)$ contains already 16 monomials.

Let us write

$$D(f) = D(a_0, \dots, a_n) = \sum_{k=(k_0,\dots,k_n)} c_k a_0^{k_0}a_1^{k_1}\cdots a_n^{k_n},$$

It is very interesting to determine the monomials $a_1^{k_1}\cdots a_n^{k_n}$ which appear in $D(f)$ with the nonzero coefficient c_k as well as the value c_k itself.

As we have seen the formula for $D(f)$ implies that $D(f)$ is a homogeneous polynomial of degree $2n-2$, i.e.,

$$D(\lambda f) = \lambda^{2n-2}D(f) \tag{2}$$

There exists another homogeneous condition for $D(f)$. If $\rho_1, \dots, \rho_n$ are roots of $g(x) = \sum_{i=0}^n a_i\lambda^i x^i$, then $\lambda\rho_1, \dots, \lambda\rho_n$ are roots of $f(x) = \sum_{i=0}^n a_ix^i$. So we obtain

$$D(g) = (\lambda^n a_n)^{2n-2}\prod_{i<j}(\rho_i - \rho_j)^2, \ \ D(f) = a_n^{2n-2}\lambda^{n(n-1)}\prod_{i<j}(\rho_i - \rho_j)^2$$

and therefore

$$D(g) = \lambda^{n(n-1)}D(f). \tag{3}$$

Two equations (2) and (3) imply that in the polynomial expression

$$D(f) = \sum_k c_k a_0^{k_0} a_1^{k_1} \cdots a_n^{k_n},$$

one has

$$\sum_{i=0}^n k_i = 2n-2 \text{ and } \sum_{i=0}^n ik_i = n(n-1)$$

for all $k = (k_0, k_1, \ldots, k_n)$ such that $c_k \neq 0$. One says that the homogeneous polynomial $D(f)$ of degree $2n-2$ is also *quasihomogeneous* of degree $n(n-1)$ if one puts $\deg a_i = i$ $(0 \leq i \leq n)$. We remark that $D(f) = D(g)$ if $g(x) = x^n f(1/x)$. Therefore $D(f)$ does not change if we replace each a_i by a_{n-i} $(i = 0, \ldots, n)$. So one can write the above two homogeneous conditions for $D(f)$ in the more symmetric equivalent way:

$$\sum_{i=0}^n ik_i = n(n-1) \text{ and } \sum_{i=0}^n (n-i)k_i = n(n-1).$$

Using these homogeneous conditions it is easy to show that $a_0^{n-1}a_n^{n-1}$ is the only possible monomial of type $a_0^i a_n^j$ which may appear in $D(f)$ with a nonzero coefficient. In order to compute the coefficient of this monomial we need to set $a_1 = \cdots = a_{n-1} = 0$ in the polynomial expression for $D(f)$. Using (1) or the formula 1 for $D(f)$ via the resultant of f and f':

$$D(f) = (-1)^{n(n-1)/2} \frac{1}{a_n} Res(f, f') = (-1)^{n(n-1)/2} n^n a_n^{n-1} \prod_{\alpha : f'(\alpha)=0} f(\alpha), \tag{4}$$

we immediately obtain

$$D(a_0 + a_n x^n) = (-1)^{n(n-1)/2} n^n a_0^{n-1} a_n^{n-1} \tag{5}$$

Definition 3.2. Newton polytope $\Delta(D(f))$ of the classical discriminant $D(f)$ is the convex hull of all $k = (k_0, k_1, \ldots, k_n) \in \mathbb{R}^{n+1}$ such that the monomial $a_0^{k_0} a_1^{k_1} \cdots a_n^{k_n}$ appears in $D(f)$ with a nonzero coefficient c_k.

We are now ready to formulate a theorem of Gelfand, Kapranov and Zelevinsky [27].

Theorem 3.3. *The Newton polytope $\Delta(D(f)) \subset \mathbb{R}^{n+1}$ is combinatorially equivalent to an $(n-1)$-dimensional cube, it contains 2^{n-1} vertices which are in a bijection with all possible subsets*

$$I \subset \{1, 2, \ldots, n-1\}.$$

If we set $i_0 := 0$, $i_{s+1} := n$, then the vertex $k(I) \in \Delta(D(f))$ corresponding to the subset $I = \{i_1 < i_2 < \cdots < i_s\}$ $(1 \leq s \leq n-1)$ has the following $n+1$ coordinates:

$$k_0 = i_1 - i_0 - 1, k_n = i_{s+1} - i_s - 1,$$

$$k_{i_p} = i_{p+1} - i_{p-1} \text{ for } i_p \in I,$$

$$k_i = 0 \text{ for } \notin I \cup \{0, n\}.$$

FIGURE 1. The Newton polytope $\Delta(D(f))$ for a cubic polynomial

If we set $l_p := i_{p+1} - i_p$ $(0 \le p \le s)$, then the monomial

$$a^{k(I)} = a_0^{l_0-1} a_{i_1}^{l_1+l_0} a_{i_2}^{l_2+l_1} \cdots a_{i_{s-1}}^{l_{s-1}+l_{s-2}} a_{i_s}^{l_s+l_{s-1}} a_n^{l_s-1}$$

appears in $D(f)$ with the coefficient

$$c_{k(I)} = \prod_{p=0}^{s} (-1)^{l_p(l_p-1)/2} l_p^{l_p}.$$

Let us illustrate the theorem in the cases $n = 2, 3$.

Example 3.4. If $n = 2$, then there exist two subsets $I \subset \{1\} : I_0 = \emptyset,\ I_1 = \{1\}$. Then $k(I_0) = (1, 0, 1)$, $k(I_1) = (0, 2, 0)$. The corresponding monomials in $D(a_0, a_1, a_2)$ are $-4a_0a_2$ and a_1^2.

Example 3.5. If $n = 3$ we obtain 4 possibilities for $I \subset \{1, 2\}$:

$$I_0 = \emptyset,\ I_1 = \{1\},\ I_2 = \{2\},\ I_3 = \{1, 2\}.$$

The corresponding monomials in $D(f)$ are

$$-27a_0^2a_3^2, -4a_1^3a_3, -4a_0a_2^3, a_1^2a_2^2.$$

The monomial $18a_0a_1a_2a_3$ corresponds to the interior lattice point $(1, 1, 1, 1) \in \Delta(D(f))$ (see Figure 1)

Let us first explain the main idea for finding vertices of $\Delta(D(f))$ and the coefficients c_k of the corresponding monomials in $D(f)$.

In order to distinguish a vertex of $\Delta(D(f))$ we consider a linear function

$$\varphi : \mathbb{R}^{n+1} \to \mathbb{R},\ \ \varphi(x) = \sum_{i=0}^{n} m_i x_i$$

and find the minimum μ of φ on $\Delta(D(f))$. We use the following basic principle from the linear programming: if φ attains the minimum μ exactly at one point $v \in \Delta(D(f))$, then v is a vertex of $\Delta(D(f))$.

Example 3.6. Consider the linear function $\varphi_0(x) := \sum_{i=1}^{n-1} x_i$. Since the monomials in $\Delta(D(f))$ correspond to lattice points with nonegative integral coordinates in $\mathbb{R}^{n+1}$, we obtain that $\varphi_0 \ge 0$ on $\Delta(D(f))$. As we have seen above the point $v_0 = (n-1, 0, \ldots, 0, n-1)$ belongs to $\Delta(D(f))$ and $\varphi_0(v_0) = 0$. Moreover, we have shown that v_0 is the only lattice point k in $\Delta(D(f))$ with $k_1 = \cdots = k_{n-1} = 0$. Therefore v_0 is a vertex of $\Delta(D(f))$. The coefficient $(-1)^{n(n-1)/2}n^n$ of the corresponding monomial $a_0^{n-1}a_n^{n-1}$ is exactly $c_{k(\emptyset)}$ as predicted by Theorem 3.3.

We will always choose the linear function φ to be $\varphi(x) := \sum_{i=0}^{n} m_i x_i$ where m_i are non-negative integers. Then

$$\min_{k \in \Delta(D(f))} \varphi(k) = \min_{c_k \neq 0} \deg(t^{\sum_i m_i k_i}),\ \ D(f) = \sum_k c_k a^k$$

FIGURE 2. The Newton diagrams of $t^{m_0}+t^{m_1}x+t^{m_2}x^2$

where $t^{\sum_i m_ik_i}$ can be obtained by the substitution $a_i := t^{m_i}$ $(i=0,\dots,n)$ into the monomial $a_0^{k_0}a_1^{k_1}\cdots a_n^{k_n}$ which appear in $D(f)$.

Assume that φ is choosen in such a way that the minimum $\mu = \min_{x\in\Delta(D(f))}\varphi(x)$ attains exactly at one vertex $k=(k_0,\dots,k_n)$ of $\Delta(D(f))$.

In this case one can compute the corresponding coefficient c_k as

$$c_k = \lim_{t\to 0}\frac{D(t^{m_0},t^{m_1},\dots,t^{m_n})}{t^\mu}. \tag{6}$$

Consider the polynomial

$$f_t(x) := t^{m_0}+t^{m_1}x+\cdots+t^{m_n}x^n.$$

We observe that the polynomial $D(t^{m_0},t^{m_1},\dots,t^{m_n})$ in the formula (6) is the classical discriminant of $f_t(x)$ with respect to the variable x.

Definition 3.7. For the polynomial

$$f_t(x) = t^{m_0}+t^{m_1}x+\cdots+t^{m_n}x^n$$

we define the **Newton diagram** $New(f_t)$ as the convex hull of $\bigcup_{i=0}^n P_i$, where P_i $(i = 0,\dots,n)$ is the vertical ray in $\mathbb{R}^2$ which consists of all points (i,y) with $y\geq m_i$. Let $\{(0,m_0),(i_1,m_{i_1}),\dots,(i_s,m_{i_s}),(n,m_n)\}$ be the set of vertices of $New(f_t)$. For our convenience we set $i_0:=0$, $i_{s+1}:=n$ and $l_p = i_{p+1}-i_p$ $(0\leq p\leq s)$. Then the rational numbers

$$\alpha_p := \frac{m_{i_{p+1}}-m_{i_p}}{l_p}\ \ (j=0,\dots,s),$$
$$\alpha_0<\alpha_1<\cdots<\alpha_s$$

are called **slopes of** $New(f_t)$. We call the polynomial $f_t(t)$ **generic** if no 3 points from the set $\{(0,m_0),(1,m_1),(2,m_2),\dots,(n,m_m)\}\subset\mathbb{R}^2$ are on the same line.

Our main tool is the following classical Newton-Puiseux theorem on the Puiseux solutions of polynomial equations:

Theorem 3.8. *Let $New(f_t)$ be the Newton diagram of a generic $f_t(x)$ as above and $\alpha_0,\alpha_1,\dots,\alpha_s$ its slopes. Then for any $p=0,\dots,s$ and for any complex root ε of the polynomial $1+t^{l_p}$ there exists a unique root $\rho(t)$ of the polynomial f_t which can be expressed as the Puiseux series*

$$\rho(t) = \varepsilon t^{-\alpha_p}(1+o(t))\in\mathbb{C}[[t^{\frac{1}{l_p}}]].$$

Example 3.9. Let us illustrate the statement of Theorem 3.8 in the case $n=2$. The polynomial $f_t(x)=t^{m_0}+t^{m_1}x+t^{m_2}x^2$ has two roots

$$\rho_{1,2}(t) = \frac{-t^{m_1}\pm\sqrt{t^{2m_1}-4t^{m_0+m_2}}}{2t^{m_2}} = \frac{-t^{m_1-m_2}}{2}\pm\sqrt{\frac{t^{2(m_1-m_2)}}{4}-t^{m_0-m_2}}.$$

$f_t(x)$ is generic if and only if $2m_1\neq m_0+m_2$. In the cases $2m_1>m_0+m_2$ and $2m_1<m_0+m_2$ we have the two possibilities for the Newton diagram $New(f_t)$ (see Figure 2).

In the case $2m_1>m_0+m_2$, one has $\alpha_{1,2}=\pm\sqrt{-1}t^{(m_0-m_2)/2}(1+o(t))$. In the case $2m_1<m_0+m_2$, one has $\rho_1(t)=-t^{m_1-m_2}(1+o(t))$ and $\rho_2(t)=-t^{m_0-m_1}(1+o(t))$. Therefore

$D(f_t) = t^{2m_2}(\rho_1(t)-\rho_2(t))^2$ is equal to $-4t^{m_0+m_2}(1+o(t))$ in the first case and to $t^{2m_1}(1+o(t))$ in the second one.

Proof of Theorem 3.3. We generalize arguments in the last example. In order to compute the maximal power μ of t which divides $D(f_t)$ for a generic $f_t(x)$ we apply Theorem 3.8 and subdivide the set $\mathcal{R} = \{\rho(t)\}$ of all roots of $f_t(x)$ into $s+1$ subsets $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_s$ ($|\mathcal{R}_p| = l_p$) such that for all $\rho(t) \in \mathcal{R}_p$ we have

$$\rho(t) = \varepsilon t^{-\alpha_p}(1 + o(t))\mathbb{C}[[t^{\frac{1}{l_p}}]]$$

for some root ε of $1 + x^{l_p}$. If we choose a root $\rho(t) \in \mathcal{R}_p$ and another root $\rho'(t)$ from $\mathcal{R}_q$ for some $q < p$, then we obtain $l_p i_p$ factors of the discriminant

$$D(f_t) = a_n^{2n-2} \prod_{i<j} (\rho_i - \rho_j)^2$$

of type $(\rho(t) - \rho'(t))^2 = \varepsilon_p^2 t^{-2\alpha_p}(1 + o(t))$ (ε_p is a root of $1 + x^{l_p}$) .

If we choose both roots $\rho(t), \rho'(t)$ in $\mathcal{R}_p$, then we obtain $l_p(l_p - 1)/2$ factors in $D(f_t)$ of type $(\rho(t) - \rho'(t))^2 = (\varepsilon_p - \varepsilon_p')^2 t^{-2\alpha_p}(1 + o(t))$, where $\varepsilon_p, \varepsilon_p'$ are two distinct roots of $1 + x^{l_p}$. This implies that the maximal power μ of t which divides $D(f_t)$ is

$$\mu = m_n(2n-2) - 2\sum_{p=1}^{s} \alpha_p l_p i_p - \sum_{p=0}^{s} l_p(l_p - 1)\alpha_p =$$

$$= m_n(2n-2) - 2\sum_{p=1}^{s} (m_{i_{p+1}} - m_{i_p}) i_p - \sum_{p=0}^{s} (m_{i_{p+1}} - m_{i_p})(l_p - 1) =$$

$$= m_0(l_0 - 1) + \sum_{p=1}^{s} m_{i_p}(l_p + l_{p-1}) + m_n(l_s - 1).$$

Since the product of squares of all complex roots of $1 + x^k$ is 1 and the product $(\varepsilon - \varepsilon')^2$ over all pairs $\{\varepsilon, \varepsilon'\}$ of roots of $1 + x^k$ is $D(1 + x^k) = (-1)^{k(k-1)/2} k^k$ we obtain that

$$\lim_{t \to 0} \frac{\Delta(t^{m_0}, t^{m_1}, \ldots, t^{m_n})}{t^{\mu}} = \prod_{p=0}^{s} (-1)^{l_p(l_p-1)/2} l_p^{l_p}$$

where $l_p = i_{p+1} - i_p$.

$\square$

Now we return to the case of arbitrary Laurent polynomials

$$f(x) = \sum_{m \in A} a_m x^m,$$

for some finite subset $A \subset M$. We consider the coefficients of f as independent variables $\{a_m\}_{m \in A}$ and denote by Δ the convex hull of A in $M_{\mathbb{R}}$.

Definition 3.10. Let

$$E_A(f) = \sum_{k \in \mathbb{Z}^A} c_k a^k$$

be the principal A-determinant of f. Then the Newton polytope of $E_A(f)$, i.e., the convex hull of lattice points $k \in \mathbb{R}^A$ such that $c_k \neq 0$, is called **secondary polytope of** A and will be denoted by $\mathrm{Sec}(A)$.

Example 3.11. If $A = \{v_0, \ldots, v_d\}$ is a set of vertices of a d-dimensional simplex Δ, then

$$E_A(f) = E_A(a_0x^{v_0} + a_1x^{v_1} + \cdots + a_dx^{v_d}) = (\mathrm{Vol}(\Delta)a_0a_1\cdots a_d)^{\mathrm{Vol}(\Delta)}.$$

In this case $\mathrm{Sec}(A)$ is a single lattice point

$$\mathrm{Vol}(\Delta)(1, \ldots, 1) \in \mathbb{R}^{d+1}.$$

Example 3.12. Let $d = 1$, $A = \{0, 1, \ldots, n\} \subset \mathbb{Z}$ and $f = a_0 + a_1x + \cdots + a_nx^n$. Since

$$E_A(f) = a_0a_nD(f)$$

the Newton polytope of $E_A(f)$ is simply a shift by $(1, 0, \ldots, 0, 1)$ of the Newton polytope of $D(f)$, we obtain that all nonzero monomials $c_k a_0^{k_0} \cdots a_n^{k_d}$ in the principal A-determinant $E_A(f)$ satisfy the following two homogenity conditions:

$$k_0 + k_1 + \cdots + k_n = 2n, \;\; k_1 + 2k_2 + \cdots nk_n = n^2.$$

By 3.3, the secondary polytope $\mathrm{Sec}(A) \subset \mathbb{R}^{n+1}$ is combinatorially equivalent to an $(n-1)$-dimensional cube, it contains 2^{n-1} vertices which are in a bijection with all possible subsets

$$I \subset \{1, 2, \ldots, n-1\}.$$

If we set $i_0 = 0$, $i_{s+1} = n$, then the vertex $k(I)$ corresponding to the subset $I = \{i_1 < i_2 < \cdots < i_s\}$ $(1 \leq s \leq n-1)$ is the sum

$$\mathrm{Sec}(\{i_0, i_1\}) + \mathrm{Sec}(\{i_1, i_2\}) + \cdots + \mathrm{Sec}(\{i_s, i_{s+1}\}).$$

If we set $l_p := i_{p+1} - i_p$ $(0 \leq p \leq s)$, i.e. l_p is the length of the interval $[i_p, i_{p+1}]$, then the monomial

$$c_{k(I)}a^{k(I)} = \left(\prod_{p=0}^{s}(-1)^{l_p(l_p-1)/2}l_p^{l_p}\right) a_0^{l_0}a_{i_1}^{l_1+l_0}a_{i_2}^{l_2+l_1}\cdots a_{i_{s-1}}^{l_{s-1}+l_{s-2}}a_{i_s}^{l_s+l_{s-1}}a_n^{l_s}$$

corresponding to $I \subset \{1, 2, \ldots, n-1\}$ can be written also in the following more elegant form

$$c_{k(I)}a^{k(I)} = \prod_{p=0}^{s} E_{\{i_p, i_{p+1}\}}(a_{i_p}x^{i_p} + a_{i_{p+1}}x^{i_{p+1}}).$$

Definition 3.13. Let $\Delta \subset M_{\mathbb{R}}$ be a d-dimensional polytope with vertices in M and A a finite subset in $\Delta \cap M$ which includes all vertices of Δ. By a **triangulation $\mathcal{T}$ of Δ associated with** A we mean a decomposition of Δ into a union of simplices $\tau_1, \ldots, \tau_k$ having vertices in A such that any nonempty intersection $\tau_i \cap \tau_j$ is a common face of τ_i and τ_j. A triangulation $\mathcal{T}$ associated with A is called **convex** if there exists a convex piecewise-linear function

$$\varphi : \Delta \to \mathbb{R}, \;\; \varphi\left(\frac{x + x'}{2}\right) \geq \frac{\varphi(x) + \varphi(x')}{2}, \;\; \forall x, x' \in \Delta$$

whose domains of linearity are precisely the d-dimensional simplices in $\mathcal{T}$.

Remark 3.14. If $\varphi : \Delta \to \mathbb{Z}$ is a convex piecewise-linear function as above, then we define a one-parameter family $f_t(x)$ of Laurent polynomials as follows:

$$f_t(x) := \sum_{m \in A} t^{\varphi(m)}a_mx^m.$$

Denote by $New(f_t)$ the **Newton diagram** of f_t. Then the projection of simplices in the low boundary $\partial New(f_t)$ of the Newton diagram are exactly simplices of the corresponding convex triangulation $\mathcal{T}$ of Δ.

Here is the main theorem of Gelfand, Kapranov and Zelevinsky about the secondary polytope (see [[28], Chapter 10]):

Theorem 3.15. *There is a natural bijection between the set of all convex triangulations $\mathcal{T}$ of Δ and the set of vertices of the secondary polytope* $\mathrm{Sec}(A) \subset \mathbb{R}^A$*:*

$$\mathcal{T} \mapsto \sum_{\tau \in \mathcal{T}(d)} \mathrm{Sec}(V(\tau)),$$

where the sum runs over all d-dimensinal simplices $\tau \in \mathcal{T}$. In this sum $V(\tau) \subset A$ denotes the set of vertices of τ and the lattice point $\mathrm{Sec}(V(\tau)) \in \mathbb{R}^{V(\tau)}$ *is considered as element of* $\mathbb{R}^A$ *with respect to the natural embedding $\mathbb{R}^{V(\tau)} \hookrightarrow \mathbb{R}^A$. If $\mathcal{T}$ an arbitrary convex triangulation of Δ, then the corresponding monomial in $E_A(f)$ has form*

$$\prod_{\tau \in \mathcal{T}(d)} E_{V(\tau)}(f_\tau),$$

where

$$f_\tau(x) = \sum_{v \in V(\tau)} a_v x^v.$$

In particular, the coefficient of this monomial equals

$$c_{\mathcal{T}} = \pm \prod_{\tau \in \mathcal{T}(d)} \mathrm{Vol}(\tau)^{\mathrm{Vol}(\tau)}.$$

It follows from this theorem, that $E_A(f)$ satifies $d+1$ homogeneity conditions. One condition we have seen already in 2.24. In order to describe d more homogeneity conditions we choose a $\mathbb{Z}$-basis $u_1, \ldots, u_d$ of the dual lattice and define a lattice vector $\beta_\Delta \in M$ by equations

$$\langle \beta_\Delta, u_i \rangle = (d+1)! \int_\Delta \langle y, u_i \rangle dy, \;\; i = 1, \ldots, d,$$

where dy is the standard Haar measure on $M_{\mathbb{R}}$ normalized by 1 on $M_{\mathbb{R}}/M$. The element $\beta \in M$ equals to the barycenter of Δ times $(d+1)\mathrm{Vol}(\Delta)$.

Assume that $\Delta = \mathrm{Conv}(A)$ is a d-dimensional polytope, where $A = \{v_1, \ldots, v_n\}$. Denote by $R(A)$ the subgroup in $\mathbb{Z}^A$ consisting of all vectors $(\lambda_1, \ldots, \lambda_n)$ such that $\sum_{i=1}^n \lambda_i v_i = 0$. Then we have the following short exact sequence

$$0 \to R(A)_{\mathbb{R}} \to \mathbb{R}^A \xrightarrow{\pi} \widetilde{M}_{\mathbb{R}} \to 0$$

Corollary 3.16. *The secondary polytope* $\mathrm{Sec}(A)$ *is a* $(|A| - d - 1)$*-dimensional polytope which lies on the affine subspace $\pi^{-1}(\widetilde{\beta}_\Delta) \subset \mathbb{R}^A$, where*

$$\widetilde{\beta}_\Delta = ((d+1)\mathrm{Vol}(\Delta), \beta_\Delta) \in \widetilde{M} = \mathbb{Z} \oplus M.$$

In the case $|A| = d+2$ the secondary polytope is 1-dimensional. It contains two vertices corresponding to 2 convex triangulations with respect to A.

Example 3.17. Take $A = \{v_0, v_1, \ldots, v_d, v_{d+1}\} \subset \mathbb{Z}^d$, where $v_1, \ldots, v_d$ is a standard basis of $\mathbb{Z}^d$, $v_0 = 0$ and $v_{d+1} = -v_1 - \cdots - v_d$. Then

$$f(x) = a_0 + a_1 x_1 + \cdots + a_d x_d + \frac{a_{d+1}}{x_1 \cdots x_d}.$$

We obtain two monomials

$$(d+1)^{d+1}(a_1 \cdots a_{d+1})^{d+1}, \;\; a_0^{d+1}(a_1 \cdots a_{d+1})^d$$

corresponding to two different convex A-triangulations of the simplex Δ. In fact, one can show that

$$E_A(f) = (a_1 \cdots a_{d+1})^d (a_0^{d+1} - (d+1)^{d+1} a_1 \cdots a_{d+1}).$$

4. Hodge-Deligne numbers

Let H be a finite-dimensional vector space over $\mathbb{Q}$. A pure Hodge structure of weight k on H is a direct sum decomposition

$$H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

such that $H^{p,q}$ is complex conjugate to $H^{q,p}$. This defines a descreasing Hodge filtration $F^{\cdot}$ on $H_{\mathbb{C}}$:

$$F^p H_{\mathbb{C}} = \bigoplus_{i \geq p} H^{i,k-i}, \;\; F^{k+1} H_{\mathbb{C}} = 0$$

such that

$$H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{r-p+1} H_{\mathbb{C}}}, \; \forall \, p \geq 0.$$

Such a structure appears in cohomology of projective algebraic manifolds [33].

A **mixed Hodge structure** on a $\mathbb{Q}$-vector space H consists of an ascending weight filtration $W_{\cdot}$ on H and a descending Hodge filtration $F^{\cdot}$ on $H_{\mathbb{C}}$ which induces a pure Hodge structure on of weight k on $\mathrm{Gr}_k^W H = W_k H / W_{k-1} H$ for all $k > 0$. In particular, one has a decomposition

$$(W_k H / W_{k-1} H)_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}.$$

The dimension of the $\mathbb{C}$-space $H^{p,q}$ is called the **Hodge-Deligne number** of H and is denoted by $h^{p,q}(H)$.

In [15] Deligne has shown that the cohomology groups $H^k(X, \mathbb{Q})$ and $H_c^k(X, \mathbb{Q})$ of any complex algebraic variety carries a natural mixed Hodge structure. For d-dimensional smooth irreducible algebraic varieties X one has a duality between $H^i(X, \mathbb{Q})$ and $H_c^{2d-i}(X, \mathbb{Q})$ $\forall i \geq 0$ defined by the Poincaré pairing

$$H^i(X, \mathbb{Q}) \times H_c^{2d-i}(X, \mathbb{Q}) \to H_c^{2d}(X, \mathbb{Q}) = \mathbb{Q}(-d),$$

where $\mathbb{Q}(-d)$ a 1-dimensional $\mathbb{Q}$-vector space of Hodge type (d, d). This duality is compatible with the Hodge structures in $H^i(X, \mathbb{Q})$ and $H_c^{2d-i}(X, \mathbb{Q})$. In particular, we have a duality between the weight filtrations

$$0 = W_{i-1} H^i(X, \mathbb{Q}) \subset W_i H^i(X, \mathbb{Q}) \subset \cdots \subset W_{2i} H^i(X, \mathbb{Q}) = H^i(X, \mathbb{Q})$$

$$0 = W_{-1} H_c^{2d-i}(X, \mathbb{Q}) \subset W_0 H_c^{2d-i}(X, \mathbb{Q}) \subset \cdots \subset W_{2d-i} H_c^{2d-i}(X, \mathbb{Q}) = H_c^{2d-i}(X, \mathbb{Q}).$$

Definition 4.1. Define a (p, q)-analog of the Euler characteristic as

$$e^{p,q}(X) := \sum_{k \geq 0} (-1)^k h^{p,q}(H_c^k(X)).$$

Since one has a long exact sequence of Hodge structures

$$\cdots \to H_c^k(X \setminus Y) \to H_c^k(X) \to H_c^k(Y) \to H_c^{k+1}(X \setminus Y) \cdots$$

for any closed subvariety $Y \subset X$, the numbers $e^{p,q}(X)$ have good behavior with respect to stratifications. If X is a disjoint union of locally closed subvarieties $X_1, \dots, X_s$, then

$$e^{p,q}(X) = \sum_{i=1}^{s} e^{p,q}(X_i).$$

It is more convenient to introduce some polynomials

$$E(X) = E(X; z, \overline{z}) := \sum_{p,q} e^{p,q}(X) z^p \overline{z}^q.$$

Then

$$E(X; z, \overline{z}) = \sum_{i=1}^{s} E(X_i; z, \overline{z})$$

if X is a disjoint union of locally closed subvarieties $X_1, \dots, X_s$. Moreover, one has

$$E(X \times X'; z, \overline{z}) = E(X; z, \overline{z}) E(X'; z, \overline{z}).$$

Example 4.2. One has the following E-polynomials for simplest algebraic varieties:

$$E(pt) = 1, \;\; E(\mathbb{P}^1) = 1 + z\overline{z}, \;\; \mathbb{A}^1 = z\overline{z}, \;\; \mathbb{A}^d = (z\overline{z})^d.$$

Since $\mathbb{T}^1 = \mathbb{A}^1 \setminus \{0\}$ we obtain

$$E(\mathbb{T}) = z\overline{z} - 1, \;\; E(\mathbb{T}^d) = (z\overline{z} - 1)^d.$$

Moreover, one has

$$H_c^i(\mathbb{T}^d) = 0, \;\; \text{if} \;\; i < d,$$
$$\dim H_c^{d+i}(\mathbb{T}^d) = h^{i,i}(H_c^{d+i}(\mathbb{T}^d)) = \binom{d}{i}, \;\; 0 \le i \le d.$$

We will be interested in cohomology of a nondegenerate affine hypersurface $Z_f \subset \mathbb{T}^d$ defined by an equation

$$f(x) = \sum_{m \in A} a_m x^m = 0.$$

On has the following Lefschetz type theorem ([17], 3.9):

Theorem 4.3. *The Gysin homomorphism*

$$H_c^i(Z_f, \mathbb{Q}) \to H^{i+2}(\mathbb{T}^d, \mathbb{Q})$$

is an isomorphism for $i > d-1$ and an epimorphism for $i = d-1$. Since Z_f is an affine smooth algbraic variety, one has $H_c^i(Z_f, \mathbb{Q})$ for $i < d-1$.

This theorem implies:

Corollary 4.4. *If $p + q > d - 1$, then $e^{p,q}(Z_f) = 0$ for $p \neq q$ and*

$$e^{p,p}(Z_f) = e^{p+1,p+1}(\mathbb{T}^d) = (-1)^{d+p+1} \binom{d}{p+1} \;\; \forall p > \frac{d-1}{2}.$$

Thus, the only interesting cohomology groups of Z_f are in degree $i = d-1$. The dimension $\dim H^{d-1}(Z_f) = \dim H_c^{d-1}(Z_f)$ can be computed using the following result of Khovansky:

Theorem 4.5. *The Euler number of a Δ-nondegenerate hypersurface $Z_f \subset \mathbb{T}^d$ equals*

$$(-1)^{d-1}\mathrm{Vol}(\Delta).$$

Corollary 4.6. *If $Z_f \subset \mathbb{T}^d$ is Δ-nondegenerate hypersurface, then one has*

$$\dim H^{d-1}(Z_f) = \dim H_c^{d-1}(Z_f) = \mathrm{Vol}(\Delta) + d - 1.$$

Using Morse theory, one can obtain the Lefschetz-type theorem for cohomology groups of Z_f over $\mathbb{Z}$ (see [43]). Using the duality and the canonical isomorphism $\Lambda^*(M) \cong H^*(\mathbb{T}^d, \mathbb{Z})$, we obtain:

Theorem 4.7. *For $i < d-1$, one has a canonical isomorphism*

$$H^i(Z_f, \mathbb{Z}) \cong \Lambda^i M.$$

In particular, one has

$$\mathrm{rk}\, H^i(Z_f, \mathbb{Z}) = \binom{d}{i}.$$

Moreover, the natural homomorphism $H^{d-1}(\mathbb{T}, \mathbb{Z}) \to H^{d-1}(Z_f, \mathbb{Z})$ is injective and it defines the exact sequence

$$0 \to \Lambda^{d-1}M \to H^{d-1}(Z_f, \mathbb{Z}) \to PH^{d-1}(Z_f, \mathbb{Z}) \to 0,$$

where

$$\mathrm{rk}\, PH^{d-1}(Z_f, \mathbb{Z}) = \mathrm{Vol}(\Delta) - 1.$$

Remark 4.8. It will be convenient to identify $PH^{d-1}(Z_f, \mathbb{Q})$ with the maximal ideal S_f^+ in the graded Artinian ring

$$S_f = S_\Delta / \langle F_0, F_1, \ldots, F_d \rangle S_\Delta$$

which we call the **Jacobian ring** of f.

It is important to stress the following general observation:

Cohomology groups of Z_f are combined from cohomology groups of the torus $\mathbb{T}^d$ and from homology groups of the Koszul complex $C.(f)$. The first ingredient is simply the exterior algebra $\Lambda^*(M)$ which does not depend on the choice of the Laurent polynomial f. The second one, i.e., the Koszul complex $C.(f)$, does depend on f. Its single nonzero homology group $H^0(C.(f)) \cong S_f$ will be of main interest.

One can show that the Hodge-Deligne number $h^{d-1,0}(H^{d-1}(Z_f))$ is a birational invariant of Z_f. In particular, one has $e^{d-1,0}(Z_f) = e^{d-1,0}(\overline{Z}_f)$ for any compactification $\overline{Z}_f$ of Z_f. An explicit formula for $e^{p,0}(Z_f)$ for all $p > 0$ has been found by Danilov and Khovansky [17]:

Theorem 4.9. *For arbitrary lattice polytope $\Delta \subset M_{\mathbb{R}}$ and for arbitrary $p > 0$, one has*

$$e^{p,0}(Z_f) = (-1)^{d-1} \sum_{\dim \theta = p+1} l^*(\theta),$$

where θ runs over the set of all $(p+1)$-dimensional faces of Δ. In particular, we obtain

$$h^{d-1,0}(H^{d-1}(Z_f)) = l^*(\Delta) = \varphi_1(\Delta).$$

Example 4.10. Let $\Delta \subset \mathbb{R}^2$ be an arbitrary 2-dimensional polytope with vertices in $\mathbb{Z}^2$ ($d = 2$). Consider a toric compactification $\overline{Z}_f$ of the affine algebraic curve $Z_f \subset \mathbb{T}^2$. Then $\overline{Z}_f$ is a smooth projective algebraic curve of genus $g = l^*(\Delta)$ and

$$E(\overline{Z}_f) = 1 - l^*(\Delta)z - l^*(\Delta)\overline{z} + z\overline{z}.$$

Let $B(\Delta)$ is the number of lattice points on the boundary $\partial\Delta$. Then $\overline{Z}_f \setminus Z_f$ consists of $B(\Delta)$ points and so we obtain

$$E(Z_f) = 1 - B(\Delta) - l^*(\Delta)z - l^*(\Delta)\overline{z} + z\overline{z}.$$

Therefore the tables of Hodge numbers for $H^1_c(Z_f)$ and $H^1(Z_f)$ have forms

$l^*(\Delta)$	0
$B(\Delta) - 1$	$l^*(\Delta)$

$l^*(\Delta)$	$B(\Delta) - 1$
0	$l^*(\Delta)$

By 4.5, we have

$$(-1)\mathrm{Vol}(\Delta) = E(Z_f; 1, 1) = 2 - B(\Delta) - 2l^*(\Delta),$$

or

$$\frac{\mathrm{Vol}(\Delta)}{2} = l^*(\Delta) - 1 + \frac{B(\Delta)}{2}.$$

The latter equation is the well-known as Pick's theorem.

There is also a general statement about $e^{0,0}(Z_f)$ [17]:

Proposition 4.11. *Let $B(\Delta)$ denotes the number of lattice points lying in the 1-dimensional skeleton of Δ. Then*

$$e^{0,0}(Z_f) = (-1)^{d-1}(B(\Delta) - 1).$$

One of the most important statement about the connection between the dimensions $\psi_i(\Delta)$ ($i > 0$) of the homogeneous components of S_f and Hodge-Deligne numbers is the following (see [17]):

Theorem 4.12. *The p-th homogeneous component $F^pH^{d-1}_c(Z_f)/F^{p+1}H^{d-1}_c(Z_f)$ of the Hodge filtration in $H^{d-1}_c(Z_f)$ has dimension*

$$h^{p,0}(H^{d-1}_c(Z_f)) + h^{p,1}(H^{d-1}_c(Z_f)) + \cdots + h^{p,d-1-p}(H^{d-1}_c(Z_f)) = \varphi_{d-p}(\Delta) = \psi_{p+1}(\Delta).$$

By duality, we obtain

Corollary 4.13. *The p-th homogeneous component $F^pH^{d-1}(Z_f)/F^{p+1}H^{d-1}(Z_f)$ of the Hodge filtration in $H^{d-1}(Z_f)$ has dimension*

$$h^{p,d-1-p}(H^{d-1}(Z_f)) + h^{p,d-p}(H^{d-1}(Z_f)) + \cdots + h^{p,d-1}(H^{d-1}(Z_f)) = \psi_{d-p}(\Delta) = \varphi_{p+1}(\Delta).$$

Example 4.14. Let $\Delta \subset \mathbb{R}^3$ be a 3-dimensional polytope. Then $Z_f \subset \mathbb{T}^3$ is an affine surface and the tables for the Hodge-Deligne numbers in $H^2_c(Z_f)$ and $H^2(Z_f)$ have forms

$l^*(\Delta)$	0	0
$\sum_\theta l^*(\theta)$	$h^{1,1}$	0
$B(\Delta) - 1$	$\sum_\theta l^*(\theta)$	$l^*(\Delta)$

$l^*(\Delta)$	$\sum_\theta l^*(\theta)$	$B(\Delta) - 1$
0	$h^{1,1}$	$\sum_\theta l^*(\theta)$
0	0	$l^*(\Delta)$

,

where

$$h^{1,1} = \varphi_2(\Delta) - \sum_\theta l^*(\theta)$$

and all sums are taken over all 2-dimensional faces $\theta \subset \Delta$.

In order to describe the cohomology group $H^{d-1}(Z_f)$ as vector space we use the surjective Poincaré residue mapping

$$H^d(\mathbb{T}^d \setminus Z_f) \stackrel{res}{\to} H^{d-1}(Z_f).$$

Denote by $\Omega^p(\mathbb{T}^d \setminus Z_f)(\log D)$ the space of algebraic differential p-forms on $D = \mathbb{T}^d \setminus Z_f$ which have at worst logarithmic singularities on $\mathbb{P}_\Delta \setminus \mathbb{T}^d$. Then the cohomology groups of the complex

$$\Omega^*(\mathbb{T}^d \setminus Z_f)(\log D)$$

are exactly $H^*(\mathbb{T}^d \setminus Z_f)$. The filtration by order of poles along Z_f determines the Hodge filtration on $H^*(\mathbb{T}^d \setminus Z_f)$ [36, 9]. A natural isomorphism

$$S_f^+ \cong \bigoplus_{i=0}^{d-1} F^i PH^{d-1}(Z_f)/F^{i+1}PH^{d-1}(Z_f)$$

is induced by the $\mathbb{C}$-linear map

$$\mathcal{L} \ : \ S_\Delta^+ \to \Omega^d(\mathbb{T}^d \setminus Z_f)(\log D)$$

which sends a monomial $x_0^k x^m$ $(m \in k\Delta \cap M)$ to the rational differential d-form

$$\frac{(-1)^{k-1}(k-1)!x^m}{f^k}\frac{dx}{x}.$$

Analogously one obtains a natural isomorphism

$$\bigoplus_{i=1}^{d} I_f^i \cong \bigoplus_{i=0}^{d-1} F^i PH_c^{d-1}(Z_f)/F^{i+1}PH_c^{d-1}(Z_f).$$

Example 4.15. Let $Z_f \subset \mathbb{T}^2$ be a Δ-nondegenerate affine algebraic curve Then the space of holomorphic differential 1-forms on $\overline{Z}_f$ can be identified with the space of 2-forms

$$\frac{x^m}{f}\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2},$$

where m is an interior lattice point in Δ. This agree with the classical description of holomorphic differential 1-forms on plane curves of degree n in $\mathbb{P}^2$ and on hyperelliptic curves.

There exist a twisted version of mixed Hodge structures associated with toric hypersurfaces $Z_f \subset \mathbb{T}^d$. A twisted version of the theorems of Danilov and Khovanski was obtained by Terasoma [**?**, 57].

Let us consider an element $\widetilde{\beta} \in \widetilde{M}_{\mathbb{Q}}$ and denote by $\widetilde{M}'$ a sublattice in $\widetilde{M}_{\mathbb{Q}}$ generated by $\widetilde{\beta}$ and $\widetilde{M}$. The dual lattice $\widetilde{N}'$ is a sublattice of finite index in $\widetilde{N}$. We can identify the lattice $\widetilde{N}$ with the homology group $H_1(\mathbb{T}^d \setminus Z_f)$. Therefore, the sublattice $\widetilde{N}' \subset \widetilde{N}$ defines a cyclic unramified Galois cover X of $\mathbb{T}^d \setminus Z_f$. The finite cyclic group $G := \widetilde{N}/\widetilde{N}'$ acts on cohomology groups of X. The G-action on $H^*(X)$ splits over $\mathbb{C}$ into a direct sum

$$H^*(X) = \bigoplus_{\chi \in G^*} H^*(X)_\chi,$$

where χ runs over all characters of G^*. The element $\widetilde{\beta} \in \widetilde{M}_{\mathbb{Q}}$ modulo $\widetilde{M}$ can be naturally identified with a particular generator $\chi_0 \in G^*$. It is easy to show that for all nontrivial elements $\chi \in G^*$ one has $H^i(X)_\chi = 0$ for all $i \neq d$ and

$$\dim H^d(X)_\chi = \mathrm{Vol}(\Delta).$$

We call $H^d(X)_{\chi_0}$ a **twisted cohomology group associated with** $\widetilde{\beta}$ and denote it for simplicity by

$$H^d(Z_f)_{\widetilde{\beta}}.$$

.

Example 4.16. Consider $\Delta = [0,n] \subset M_{\mathbb{R}} = \mathbb{R}$, where $n = 2g$ is an even number. We identify $\widetilde{M}$ with $\mathbb{Z}^2$ and take $\widetilde{\beta} = (1/2, 1/2) \in \widetilde{M}_{\mathbb{Q}}$. The element $\widetilde{\beta}$ defines a double cover X of $\mathbb{T}^1 \setminus Z_f$ (i.e., $\mathbb{C}^*$ minus $2g$ points) and the twisted cohomology group $H^1(X)_{\chi_0}$ ($\dim H^1(X)_{\chi_0} = 2g$) can be naturally identified with the cohomology group of the hyperelliptic curve $\overline{X}$ of genus g.

For computing Hodge numbers in the twisted cohomology group $H^d(Z_f)_{\widetilde{\beta}}$ Terasoma introduces the notion of *twisted Ehrhart polynomial*.

Definition 4.17. Denote by $\widetilde{\Delta}$ the convex hull of $(1,\Delta) \subset \widetilde{M}$ and $0 \in \widetilde{M}$. Take an element $\widetilde{\beta} \in \widetilde{M}_{\mathbb{R}}$ and define the **twisted Ehrhart polynomial** $\Psi(\Delta, t, \widetilde{\beta})$ by

$$\Psi(\Delta, t, \widetilde{\beta}) = (1-t)^{d+2} \sum_{k \geq 0} l(k\widetilde{\Delta}, \widetilde{\beta}),$$

where $l(k\widetilde{\Delta}, \widetilde{\beta})$ denotes the number of points in $\widetilde{\beta} + \widetilde{M}$ which are contained in $k\widetilde{\Delta}$. If $\widetilde{\beta} \in \widetilde{M}$, then we come to our previous definition since in this case we have

$$l(k\widetilde{\Delta}, 0) = \sum_{i=0}^{k} l(i\Delta).$$

Analogously, we define the polynomial

$$\Phi(\Delta, t, \widetilde{\beta}) = (1-t)^{d+2} \sum_{k \geq 0} l^*(k\widetilde{\Delta}, \widetilde{\beta}),$$

where $l^*(k\widetilde{\Delta}, \widetilde{\beta})$ denotes the number of points in $\widetilde{\beta} + \widetilde{M}$ which are contained in the interior of $k\widetilde{\Delta}$.

Definition 4.18. An element $\widetilde{\beta} \in \widetilde{M}_{\mathbb{R}}$ is called **nonresonant** if

$$l(k\widetilde{\Delta}, \widetilde{\beta}) = l^*(k\widetilde{\Delta}, \widetilde{\beta})$$

holds for all $k \geq 0$.

It is easy to see that nonresonant elements form an open dense subset in $M_{\mathbb{R}}/M$.

Example 4.19. Let $\Delta = [0,n] \subset \mathbb{R}$ and $\widetilde{\Delta} = \mathrm{Conv}((0,0),(1,0),(1,n))$. Then the set of nonresonant elements in $M_{\mathbb{R}}/M$ consists of $n+1$ connected components parametrized by the following possibilities for the twisted Ehrhart polynomial

$$\Psi_k(\Delta, t, \widetilde{\beta}) = k + (n-k)t, \quad 0 \leq k \leq n.$$

There exists the following twisted version of the Ehrhart reciprocity law:

Theorem 4.20. *Let $\Delta \subset M_{\mathbb{R}}$ be an arbitrary d-dimensional polytope with vertices in M. Then for any element $\widetilde{\beta} \in \widetilde{M}_{\mathbb{R}}$ one has*

$$\Phi(\Delta, t, \widetilde{\beta}) = t^{d+1} \Psi(\Delta, t^{-1}, -\widetilde{\beta}).$$

We remark that instead of $Z_f \subset \mathbb{T}^d$ one can consider the family of equations

$$f(x) - t = 0,$$

where t is sufficiently large complex number. In this way we obtain a 1-parameter family of affine hypersurfaces $Z_{f_t} \subset \mathbb{T}^d$ which are fibers of a locally trivial fibration defined by $f : \mathbb{T}^d \to \mathbb{C}$ over the open set $U_r := \{t \in \mathbb{C} \ : \ |t| > r\}$ which does not contain any critical value of f.

The cohomology $H^*(Z_{f_t}$ together with the natural action of the monodromy around $t = \infty$ can be described in terms of the lattice polytope $\Delta_\infty(f) := \mathrm{Conv}(\{0\} \cup A)$ which is called **Newton polytope of f at infinity**.

Definition 4.21. [37] A Laurent polynomial f is called to be **nondegenerate with respect to** $\Delta_\infty(f)$ if for any face $\theta \subset \Delta_\infty(f)$ which does not contain 0 the polynomial system

$$x_1 \frac{\partial}{\partial x_1} f_\theta(x) = \cdots = x_d \frac{\partial}{\partial x_d} f_\theta(x) = 0$$

has no solutions in $\mathbb{T}^d$ for any extension of the base field K.

If f is nondegenerate with respect to $\Delta_\infty(f)$ and $\dim \Delta_\infty(f) = d$, then the Euler number of Z_{f_t} $(t \in U_r)$ is equal to

$$(-1)^{d-1}\mathrm{Vol}(\Delta_\infty).$$

If θ is face of Δ_∞ of codimension one which does not contain 0, then there exists a unique primitive lattice point $u_\theta \in N$ such that θ is the intersection of Δ_∞ with the affine hyperplane $\langle *, u_\theta \rangle = \mu_\theta$ for some positive integer μ_θ.

The following theorem is due to Libgober and Sperber [39].

Theorem 4.22. *Assume that f is nondegenerate with respect to $\Delta_\infty(f)$ and $\dim \Delta_\infty(f) = d$, then zeta function of monodromy at $t = \infty$ equals*

$$Z(f_t, s) = \prod_{\dim \theta = d-1} (1 - s^{\mu_\theta})^{(-1)^{d-1}\mathrm{Vol}(\theta)},$$

where the product is taked over all faces θ of Δ_∞ which do not contain 0. We note that

$$\prod_{\dim \theta = d-1} \mu_\theta \mathrm{Vol}(\theta) = \mathrm{Vol}(\Delta_\infty).$$

5. GKZ-hypergeometric system

Let $A = \{v_1, \ldots, v_n\}$ be a finite subset in $M \cong \mathbb{Z}^d$.

In order to motivate the definition of the generalized hypergeometric system introduced by Gelfand, Kapranov and Zelevinsky we start with the consideration of the integrals

$$I(\alpha, \beta) := \int_\gamma x^{-\beta} \exp\left(\sum_{i=1}^n a_i x^{v_i}\right) \frac{dx}{x},$$

where β is an element of $M_\mathbb{C} = M \otimes_\mathbb{Z} \mathbb{C}$ and γ is a d-dimensional cycle in $\mathbb{T} \cong (\mathbb{C}^*)^d$ such that $\partial\gamma = 0$.

Example 5.1. A simplest example of such an integral for $d = 1$ is the Hankel formula for the classical Γ-function

$$\frac{1}{\Gamma(\beta+1)} = \frac{1}{2\pi i}\int_C z^{-\beta}\exp(z)\frac{dz}{z}, \quad \operatorname{Re}\beta < -1$$

where C goes along the boundary of the cut of $\mathbb{C}$ along the real ray from $-\infty$ to 0. More generally, one has

$$\frac{1}{2\pi i}\int_C z^{-\beta}\exp(az)\frac{dz}{z} = \frac{a^\beta}{\Gamma(\beta+1)}.$$

By the formal differentiation under the integral sign we obtain

$$\frac{\partial}{\partial a_j} I(a,\beta) = I(a, \beta + v_j) \;\; \forall \;\; 1 \le j \le n.$$

If $l = (l_1, \ldots, l_n) \in R(A)$, then $\sum_{i=1}^n l_i v_i = 0$, i.e.,

$$\sum_{l_i > 0} l_i v_i = \sum_{l_j<0} (-l_j) v_j,$$

so we have

$$\prod_{l_i>0}\left(\frac{\partial}{\partial \alpha_i}\right)^{l_i} I(a,\beta) = I(a, \beta + \sum_{l_i>0} l_i v_i) = I(a, \beta + \sum_{l_j<0}(-l_j)v_j) = \prod_{l_j<0}\left(\frac{\partial}{\partial a_j}\right)^{-l_j} I(a,\beta).$$

If we introduce the differential operator

$$\Box_l := \prod_{l_i>0}\left(\frac{\partial}{\partial a_i}\right)^{l_i} - \prod_{l_j<0}\left(\frac{\partial}{\partial a_j}\right)^{-l_j},$$

then we obtain

$$\Box_l I(a,\beta) = 0 \;\; \forall \, l \in R(A).$$

A choice of a $\mathbb{Z}$-basis $u_1, \ldots, u_d$ of N determines an isomorphism $\mathbb{T} \cong (\mathbb{C}^*)^d$ and the coordinates $x_1, \ldots, x_d$ on $\mathbb{T}$ together with d vector fields on $\mathbb{T}$

$$x_1\frac{\partial}{\partial x_1}, \ldots, x_d\frac{\partial}{\partial x_d}$$

such that

$$x_k \frac{\partial}{\partial x_k} x^m = \langle m, u_k\rangle x^m, \;\; 1 \le k \le d.$$

We define d first order differential operators

$$D_k^\beta := \sum_{i=1}^n \langle v_i, u_k\rangle a_i \frac{\partial}{\partial a_i} - \langle \beta, u_k\rangle, \;\; 1 \le k \le d.$$

Then for all $1 \le k \le d$ we have

$$D_k^\beta I(a,\beta) = \int_\gamma x_k \frac{\partial}{\partial x_k}\left(x^{-\beta}\exp\left(\sum_{i=1}^n a_i x^{v_i}\right)\right)\frac{dx}{x} = 0.$$

Definition 5.2. [26] The GKZ-hypergeometric system with the parameter $\beta \in M_{\mathbb{C}}$ is the system of differential equations for functions $\Phi(a)$

$$\Box_l \Phi(a) = 0, \ \forall\, l \in R(A),$$
$$D_k^{\beta} \Phi(a) = 0, \ 1 \leq k \leq d.$$

Remark 5.3. The first part of equations in the GKZ-hypergeometric system consists of infinitely many equations parametrized by elements l of the lattice $R(A)$. One can show that this system is equivalent to a finite system

$$\Box_l \Phi(a) = 0, \ \ l \in B$$

for some finite subset B in $R(A)$. However, a priori there is no canonical choice of the subset $B \subset R(l)$.

The differential operators D_k^{β} $(1 \leq k \leq d)$ depend on the choice of a $\mathbb{Z}$-basis $u_1, \ldots, u_d$ of N, but solutions of the system

$$D_k^{\beta} \Phi(a) = 0, \ 1 \leq k \leq d$$

are independent of this basis.

Example 5.4. Let $d = 2$, $n = 3$, $v_1 = (1,0)$, $v_2 = (1,1)$, $v_3 = (1,2)$. Then the lattice $R(A)$ is generated by $(1,-2,1)$ $(v_1 + v_3 = 2v_2)$. The GKZ-hypergeometric system reduces to the following 3 equations

$$\frac{\partial}{\partial a_1} \frac{\partial}{\partial a_3} \Phi = \left(\frac{\partial}{\partial a_2} \right)^2 \Phi,$$
$$\left(a_1 \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2} + a_3 \frac{\partial}{\partial a_3} - \beta_1 \right) \Phi = 0,$$
$$\left(a_2 \frac{\partial}{\partial a_2} + 2a_3 \frac{\partial}{\partial a_3} - \beta_2 \right) \Phi = 0.$$

Theorem 5.5. *Let $d = 2$, $M = \mathbb{Z}^2$ and A be the set of $n+1$ vectors*

$$v_0 = (1,0), v_1 = (1,1), \ v_2 = (1,2), \ldots, v_n = (1,n).$$

Denote by $\rho_1, \ldots, \rho_n$ the roots of a general equation

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$$

of degree n. Then for any nonzero complex number μ the functions

$$\rho_1^{\mu}, \ldots, \rho_n^{\mu}$$

form a basis for the solution space of the A-hypergeometric system with the parameter $\beta = (0, -\mu)$.

Proof. Consider the integral

$$I(f,\gamma) = \frac{1}{2\pi i} \int_{\gamma} z^{\mu} d \ln(f(z)) = \frac{1}{2\pi i} \int_{\gamma} z^{\mu} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{\gamma} z^{\mu} \left(\sum_{i=1}^{n} \frac{1}{z - \rho_i} \right) dz.$$

If γ is choosen as small circle containing ρ_i and no other roots ρ_j $j \neq i$, then $I(f,\gamma)$ equals ρ_i^{μ}. On the other hand,

$$\frac{\partial}{\partial a_k} I(f,\gamma) = \frac{1}{2\pi i} \int_{\gamma} z^{\mu} \left(\frac{k z^{k-1}}{f(z)} - \frac{z^k f'(z)}{f^2(z)} \right) dz$$

$$\frac{\partial}{\partial a_l}\frac{\partial}{\partial a_k}I(f,\gamma) = \frac{1}{2\pi i}\int_\gamma z^\mu\left(\frac{-kz^{l+k-1}f'(z)}{f^2(z)} - \frac{lz^{k+l-1}f'(z)}{f^2(z)} + 2\frac{z^{k+l}(f'(z))^2}{f^3(z)}\right)dz.$$

Therefore, we have

$$\frac{\partial}{\partial a_l}\frac{\partial}{\partial a_k}I(f,\gamma) = \frac{\partial}{\partial a_i}\frac{\partial}{\partial a_j}I(f,\gamma)$$

if $k+l = i+j$. Two homogeneity conditions are abviously satisfied, because f and λf have the same roots and roots of $f(\lambda x)$ are obtained from roots of f through division by λ.

□

Corollary 5.6. (Mayer) [45] *The roots $\rho_1, \ldots, \rho_n$ of a general equation*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$$

of degree n form a basis for the solution space of the A-hypergeometric system with the parameter $\beta = (0,-1)$.

Example 5.7. Let ρ_1, ρ_2 be two roots of $f(x) = a_0 + a_1 x + a_2 x^2$.

$$\rho_1 = -\frac{a_0}{a_1}\sum_{m=0}^{\infty}\frac{1}{m+1}\binom{2m}{m}\left(\frac{a_0 a_2}{a_1^2}\right)^m$$

$$\rho_2 = -\frac{a_1}{a_2} + \frac{a_0}{a_1}\sum_{m=0}^{\infty}\frac{1}{m+1}\binom{2m}{m}\left(\frac{a_0 a_2}{a_1^2}\right)^m$$

For small values of $z = \frac{a_0 a_2}{a_1^2}$ the ratio ρ_1/ρ_2 is small

Applying the same arguments to $\log z$ instead of z^μ.

Theorem 5.8. (Skarke) [52] *If one considers the previous A-hypergeometric system with the parameter $\beta = (0,0)$, then*

$$1, \ln(\rho_1/\rho_2), \ldots, \ln(\rho_{n-1}/\rho_n)$$

form a basis of the solution space.

Remark 5.9. All 2^{n-1} distinct Puiseux series for the roots $\rho_1, \ldots, \rho_n$ were found by Sturmfels ([55], Theorem 3.2).

Example 5.10. The integral

$$I(a) = \int_\gamma \frac{dz}{\sqrt{z(a_1 + a_2 z + a_3 z^2)}} = \int_\gamma \frac{dz}{\sqrt{a_1 z + a_2 z^2 + a_3 z^3}}.$$

is a solution of the GKZ-hypergeometric system for

$$A = \{v_0 = (1,0), v_1 = (1,1), v_2 = (1,2)\}, \quad \beta = (-1/2, 1/2).$$

Indeed we have

$$\frac{\partial}{\partial a_1}\frac{\partial}{\partial a_3}I(a) = \left(\frac{\partial}{\partial a_2}\right)^2 I(a) = \frac{3}{4}\int_\gamma \frac{z^{3/2}dz}{(a_1 z + a_2 z^2 + a_3 z^3)^{5/2}}$$

$$\left(a_1\frac{\partial}{\partial a_1} + a_2\frac{\partial}{\partial a_2} + a_3\frac{\partial}{\partial a_3}\right)I(a) = -\frac{1}{2}\int_\gamma \frac{(a_1 + a_2 z + a_3 z^2)dz}{\sqrt{z(a_1 + a_2 z + a_3 z^2)^3}} = -\frac{1}{2}I(a),$$

$$\left(a_1\frac{\partial}{\partial a_1} + 2a_2\frac{\partial}{\partial a_2} + 3a_3\frac{\partial}{\partial a_3}\right)I(a) = -\frac{1}{2}\int_\gamma \frac{z(a_1 + 2a_2 z + 3a_3 z^2)dz}{\sqrt{(za_1 + a_2 z^2 + a_3 z^3)^3}} = 0.$$

The integral $I(a)$ describes periods of the elliptic curve

$$w^2 = z(a_1 + a_2 z + a_3 z^2), \;\; a_1 a_3 (a_2^2 - 4a_1 a_2) \neq 0.$$

Remark 5.11. In [26] the finite subset $A = \{v_1, \ldots, v_n\} \subset M$ was assumed to generate M and satisfy $\langle v_i, n_A \rangle = 1$ $(1 \leq i \leq n)$ for some element n_A of the dual lattice N. The last condition imply that the GKZ-hypergeometric system has only **regular singularties** and its solutions are generalized hypergeometric functions of nonconfluent type. The singular locus of this A-hypergeometric system is described by the condition $E_A(f) = 0$. A more general situation in which vectors $v_1, \ldots, v_n$ need not to be on an affine hyperplane was considered by Dwork-Loeser [22, 23] and Adolphson [1]. In general, one obtains a holonomic system whose solutions are generalized hypergeometric functions of confluent type.

Example 5.12. Let $A = \{-1, 1\} \subset \mathbb{Z} = M$. Then

$$F(\beta + 1, a_1, a_2) := \frac{1}{2\pi i} \int_C \exp\left(a_1 z + \frac{a_2}{z}\right) \frac{dz}{z} = (\alpha_1)^\beta \, {}_0F_1(\beta + 1, (a_1 a_2)),$$

where

$${}_0F_1(\lambda, t) := \sum_{k=0}^{\infty} \frac{\Gamma(\lambda)}{k!\Gamma(\lambda + k)} t^k = \frac{1}{2\pi i} \int_C z^{-\lambda} \exp\left(z + \frac{t}{z}\right) dz$$

is the classical confluent hypergeometric series. We remark that

$$J_0(x) = \frac{1}{2\pi i} \int_C \exp\left(z - \frac{x^2}{4z}\right) \frac{dz}{z}$$

is the classical Bessel function.

Definition 5.13. Let $\Delta_\infty = \Delta_\infty(A)$ be the convex hull in $M_{\mathbb{R}}$ of 0 and $A = \{v_1, \ldots, v_n\}$. If

$$f(x) = \sum_{m \in A} a_m x^m,$$

then we call Δ_∞ the **Newton polytope of f at infinity**.

Remark 5.14. If all elements of A belong to an affine hyperplane $\langle *, n_A \rangle = 1$ for some element n_A of the dual lattice N, then Δ_∞ can be identified with the polytope $\widetilde{\Delta}$ from our previus consideration, where Δ is the convex hull of A (we assume that $\dim \Delta = d - 1$).

One can prove that GKZ-hypergeometric system is holonomic and its space of solutions has dimension $d!\mathrm{Vol}(\Delta_\infty)$. In the regular case, one can sometimes constract explicitly a basis for solutions. For this purpose we introduce some new definitions.

Definition 5.15. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an arbitrary element of $\mathbb{C}^n$. We consider the formal series

$$\Phi_\lambda(a) = \sum_{l \in R(A)} \frac{1}{(2\pi i)^n} \int_{C^n} z^{-\lambda - l} \exp\left(\sum_{i=1}^n a_i z_i\right) \frac{dz}{z} =$$

$$= \sum_{l \in R(A)} \left(\prod_{i=1}^n \Gamma(\lambda_i + l_i + 1)^{-1}\right) a^{\lambda + l}$$

Let $e_1, \ldots, e_n$ be a standard basis of $\mathbb{C}^n$. Using the property $\Gamma(z+1) = z\Gamma(z)$, we obtain

$$\frac{\partial}{\partial a_j}\Phi_\lambda(a) = \Phi_{\lambda - e_j}(a), \;\; 1 \le j \le n.$$

It follows immediately from the definition $\Phi_\lambda(\alpha)$ that

$$\Phi_\lambda(a) = \Phi_{\lambda'}(a), \;\text{ if }\; \lambda - \lambda' \in R(A).$$

These properties imply that

$$\Box_l \Phi_\lambda(a) = 0, \; \forall\, l \in R(A).$$

On the other hand, for any $l = (l_1, \ldots, l_n) \in R(A)$ we have

$$\left(\sum_{i=1}^n \langle v_i, u_k\rangle a_i \frac{\partial}{\partial a_i}\right) a^{\lambda+l} = \left(\sum_{i=1}^n (\lambda_i + l_i)\langle v_i, u_k\rangle\right) a^{\lambda+l} =$$

$$= \langle \sum_{i=1}^n \lambda_i v_i, u_k\rangle a^{\lambda+l} + \langle \sum_{i=1}^n l_i v_i, u_k\rangle a^{\lambda+l} = \langle \sum_{i=1}^n \lambda_i v_i, u_k\rangle a^{\lambda+l},$$

since $\sum_{i=1}^n l_i v_i = 0$. This shows that $\Phi_\lambda(a)$ formally satisfies the GKZ-system with the parameter $\beta = \sum_{i=1} \lambda_i v_i \in M_{\mathbb{C}}$.

Definition 5.16. A subset $I \subset \{1, \ldots, n\}$ such that $|I| = d$ is called a base if the element $\{v_i\}_{i \in I} \subset M$ a linearly independent, i.e. they form a basis of $M_{\mathbb{Q}}$.

Every base $I \subset \{1, \ldots, n\}$ defines a $\mathbb{Q}$-splitting of the exact sequence

$$0 \to R(A) \to \mathbb{Z}^n \to M \to 0$$

Using the isomorphisms

$$\mathbb{Q}^I \cong M_{\mathbb{Q}}, \;\; \mathbb{Q}^{\overline{I}} \cong R(A)_{\mathbb{Q}},$$

where $\overline{I} = \{1, \ldots, n\} \setminus I$, and the standard lattices $\mathbb{Z}^I \subset \mathbb{Q}^I$, $\mathbb{Z}^{\overline{I}} \subset \mathbb{Q}^{\overline{I}}$, we obtain a sublattice M_I of finite index $r(I)$ in M generated by $\{v_i\}_{i \in I}$ and a lattice $\Lambda(I) \subset R(A)_{\mathbb{Q}}$ containing $R(A)$ together with a natural isomorphism

$$M/M_I \cong \Lambda(I)/R(A).$$

Using the splitting of

$$0 \to R(A)_{\mathbb{C}} \to \mathbb{C}^n \to M_{\mathbb{C}} \to 0$$

defined by a base I, we can define the canonical i-lifting β_I of an element $\beta \in M_{\mathbb{C}}$.

Let $\delta_1, \ldots, \delta_{r(I)} \in \Lambda(I)$ be representatives of all elements in the finite group $\Lambda(I)/R(A)$. Then $r(I)$ power series

$$\Phi_{\delta_j - \beta_I}(\alpha), \;\; 1 \le j \le r(I)$$

are solutions of the GKZ-system with the parameter β. Moreover, they can be written as

$$a^{\delta_j + \beta_I} \sum_{l \in R(A) \cap C_j(\beta)} c_l a^l.$$

for some $(n-d)$ dimensional cone $C_j(\beta)$ in $R(A)_{\mathbb{R}}$.

Definition 5.17. Let $\mathcal{T}$ be a convex triangulation of Δ. An element $\beta \in M_{\mathbb{C}}$ is called $\mathcal{T}$-nonresonant if the lattices $\beta_I + \Lambda(I)$ and $\beta_{I'} + \Lambda(I')$ have empty intersection for any two different d-dimensional simplices $I, I' \in \mathcal{T}$.

Theorem 5.18. [26] *Assume that A belong to an affine hyperplane $\langle *, n_A \rangle = 1$ for some element n_A of the dual lattice N. Let $\mathcal{T}$ be a convex triangulation of $\Delta = \mathrm{Conv}(A)$ corresponding to a vertex of the secondary polytope $\mathrm{Sec}(A)$. If β is $\mathcal{T}$-nonresonant, then the power series*

$$\Phi_{\delta_j+\beta_I}(\alpha), \ \ 1 \leq j \leq r(I), \ \ I \in \mathcal{T}$$

have common convergency domain and form a $\mathbb{C}$-basis of the space of solutions which has dimension $\mathrm{Vol}(\Delta) = \mathrm{Vol}(\Delta_\infty)$.

Example 5.19. Let Δ be a simplex. Consider a convex triangulation $\mathcal{T}$ of Δ consisting of the single simplex Δ. Then every $\beta \in M_{\mathbb{C}}$ is $\mathcal{T}$-nonresonance. So we can find a basis of solutions for any β. Take $\beta = 0$ for our previous example. The lattice $\Lambda(I)$ is generated by $(1/2, -1, 1/2)$. We obtain two representatives for $\Lambda(I)/R(A)$: $\delta_1 = (0, 0, 0)$ and $\delta_2 = (1/2, -1, 1/2)$. Then two independent solutions are

$$\Phi_{\delta_1} = \sum_{l \in \mathbb{Z}} \Gamma(l+1)^{-2}\Gamma(-2l+1)^{-1}\frac{a_1^l a_3^l}{(a_2)^2 l} = 1,$$

$$\Phi_{\delta_2} = \left(\frac{a_1a_3}{(a_2)^2}\right)^{1/2} \sum_{l \in \mathbb{Z}} \Gamma(l+1+1/2)^{-2}\Gamma(-2l)^{-1}\frac{a_1^l a_3^l}{(a_2)^{2l}} = \sum_{l<0} \Gamma(l+1+1/2)^{-2}\Gamma(-2l)^{-1}\frac{a_1^l a_3^l}{(a_2)^{2l}}$$

Definition 5.20. Let $\theta_1, \ldots, \theta_s$ be the codimension-1 faces of Δ that contain the origin $0 \in M$. Each θ_i is the intersection of the hyperplane defined by the equation $\langle *, l_i \rangle = 0$ with Δ, where l_i is a primitive element of N such that $\langle *, l_i \rangle \geq 0$ on Δ. Am element $\beta \in M_{\mathbb{C}}$ is called **semi-nonresonant** if $\langle *, l_i \rangle$ is not 0 or a negative integer for all $i = 1, \ldots, s$. Am element $\beta \in M_{\mathbb{C}}$ is called **nonresonant**, if $\langle \beta, l_i \rangle \notin \mathbb{Z}$ for all $i = 1, \ldots, s$

The following result was proved by Adolphson [1]:

Theorem 5.21. *Assume that A generates M and β is a semi-nonresonant. Then the solution of the A-hypergeometric system form a vector space of dimension $\mathrm{Vol}(\Delta_\infty)$.*

Remark 5.22. For arbitrary β one has $\mathrm{Vol}(\Delta_\infty)$ only as a low estimate for the dimension of the solution space [50, 51]. The resonant case $\beta = 0$ is especially interesting from view-point of toric mirror symmetry [54].

6. Exponential sums and the method of Dwork

Let p be a prime number, $q = p^r$, and $\mathbb{F}_q$ the finite field with q elements. Consider a Laurent polynomial

$$f(x) = \sum_{m \in A} a_m x^m \in \mathbb{F}_q[M] \cong \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$$

as a regular function on the d-dimensional algebraic torus $\mathbb{T}$ and choose a nontrivial additive character $\Psi \colon \mathbb{F}_q \to \mathbb{C}^\star$. The polynomial f and the character Ψ determine for any $k \geq 1$ the exponential sum

$$S_k(\Psi, f) := \sum_{x_1, \ldots, x_d \in \mathbb{F}_{q^k}^*} \Psi\left(\mathrm{Tr}_{\mathbb{F}_{q^s}/\mathbb{F}_q} f(x_1, \ldots, x_d)\right)$$

and the corresponding L-function

$$L(\Psi, f, t) = \exp\left(\sum_{k=1}^{\infty} S_k(\Psi, f)\frac{t^k}{k}\right).$$

Dwork attached to f and Ψ a complex $DK^\bullet$, of p-adic Banach spaces, so called **p-adic Dwork-Koszul complex**, such that its cohomology groups $H^i(DK^\bullet)$ are vector spaces over a finite extension of $\mathbf{Q}_p$ endowed with a Frobenius automorphism F, and one has

$$L(\Psi, f, t) = \prod_{i=0}^{n} \det(\mathrm{Id} - tF|H^i(DK^\bullet)^{(-1)^{i+1}}.$$

The following result is due to Adolphson and Sperber [2]:

Theorem 6.1. *If f is nondegenerate with respect to $\Delta_\infty(f)$ and $\dim \Delta_\infty(f) = d$, then*

$$L(\Psi, f, t)^{(-1)^{d-1}}$$

is a polynomial of degree $\mathrm{Vol}(\Delta_\infty(f))$, where $\Delta_\infty(f)$ is convex hull of 0 and all lattice points which occur as exponents in monomials of f.

Remark 6.2. If A is contained in affine hyperplane $\langle *, u\rangle$ for some $u \in N$ and $\dim \Delta_\infty(f) = d$, then $DK^\bullet$ is a p-adic completion of the Koszul complex $C_\bullet(f)$ considered in section 2. In this case, we have $\mathrm{Vol}(\Delta_\infty(f)) = \mathrm{Vol}(\Delta)$, where Δ is the convex hull of A. Moreover, the L-function $L(\Psi, f, t)$ is equal to nontrivial factor in the the Weil zeta-function of the affine $(d-2)$-dimensional hypersurface $Z_f \subset \mathbb{T}^{d-1}$ (see [20, 21, 38]).

Example 6.3. Let $\varepsilon \in \mathbb{C}^*$ be a primitive p-th root of unity. One defines an additive character $\mathbb{F}_p \to \mathbb{C}^*$ using the homomorphism

$$\mathbb{Z} \to \mathbb{C}^*, \quad a \mapsto \varepsilon^a.$$

The integral is analogous to the sum

$$\sum_{x_0, x_1, \ldots, x_n \in \mathbb{F}_p^*} \varepsilon^{x_0 f(x)} = pN'_p - (p-1)^d,$$

where N'_p is the number of solutions of the equation $f(x) = 0$ in $(\mathbb{F}_p^*)^d$.

Let us discuss combinatorial aspects of the complex $DK^\bullet$.

We denote by $C(\Delta_\infty)$ cone the union of all rays in $M_\mathbb{R}$ emanating from the origin and passing through $\Delta_\infty(f)$. Define a piecewise linear **weight function**:

$$w \;:\; C(\Delta_\infty) \to \mathbb{R}$$

$$\alpha \mapsto w(\alpha)$$

where $w(\alpha)$ is the smallest positive real number λ such that $\alpha \in \lambda\Delta_\infty(f)$. One has the following properties of the weight function:

(1) $w(\lambda\alpha) = \lambda w(\alpha) \;\; \forall \lambda \in \mathbb{R}_{>0}$,
(2) $w(\alpha) = 0 \;\Leftrightarrow\; \alpha = 0$,
(3) $w(\alpha_1 + \alpha_2) \leq w(\alpha_1) + w(\alpha_2)$ and the equality holds if and only if the rays from 0 to α_1 and α_2,
(4) $w(M \cap C(f))$ is contained in $(1/k)\mathbb{Z}$ for some positive integer k.

The weight function defines an increasing filtration on the ring $K[M \cap C(\Delta_\infty)]$ by finite dimensional subspaces generated by monomials of weight $\leq i/k$ for some positive $i \in \mathbb{Z}$. We denote by $S(\Delta_\infty)$ the associated graded ring (it is graded by nonnegative rational numbers from $(1/k)\mathbb{Z}$). Since the Laurent polynomial f has weight 1 it defines an element $F \in S^1(\Delta_\infty)$. Denote

$$F_i := x_i \frac{\partial F}{\partial x_i}, \quad i = 1, \dots, d.$$

If f is nondegenerate with respect to $\Delta_\infty(f)$ and $\dim \Delta_\infty(f) = d$, then $F_1, \dots, F_d$ form a regular sequence in $S(\Delta_\infty)$ [37]. The Koszul complex associated with $F_1, \dots, F_d$ is acyclic except in degree 0. One can show that the dimension of the Artinian ring

$$S(\Delta_\infty)/\langle F_1, \dots, F_d \rangle S(\Delta_\infty)$$

equals $\mathrm{Vol}(\Delta_\infty)$. This ring is analogous to the Jacobian ring S_f in our previous considerations.

Remark 6.4. There exists a twisted version of exponential sums. Given a k-regular function f on $(\mathbb{T})^d$, a nontrivial additive character $\Psi \colon \mathbb{F}_q \to \mathbb{C}^*$, and a multiplicative character $\chi \in M_{\mathbb{Q}}$ $\chi \colon \mathbb{T}^d(\mathbb{F}_q) \to \mathbb{C}^*$, one can form exponential sums

$$S_k(\Psi, \chi, f) = \sum_{x \in \mathbb{T}^d(\mathbb{F}_{q^k})} (\chi \circ \mathrm{N}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(x))(\Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_q}(f(x)))$$

(where k_m is the extension of k of degree m) and an associated L-function

$$L(\Psi, \chi, f; t) = \exp\left(\sum_{k=1}^{\infty} S_k(\Psi, \chi, f) \frac{t^k}{k} \right).$$

For general results see [4, 5].

Example 6.5. If ψ is nontrivial additive character of $\mathbb{F}_q$, then

$$S(q,b) := \sum_{x \in \mathbb{F}_q^*} \psi\left(t + \frac{b}{t}\right) = 2\sqrt{q}\cos\theta(q,b) \;\; (0 \leq \theta(q,b) < \pi)$$

is a classical **Kloosterman sum**. If χ is a multiplicative character of $\mathbb{F}_q^*$, then

$$K(b, \chi, \psi) := \sum_{x \in \mathbb{F}_q^*} \chi(t)\psi\left(t + \frac{b}{t}\right)$$

is called **twisted Kloosterman sum**.

7. Topology of real and complex toric hypersurfaces

Let $K = \mathbb{R}$, or $K = \mathbb{C}$. We are interested in topology of the set $Z_f(K)$ of solutions of the equation $f(x) = 0$ in $\mathbb{T}^d(K)$ ($\cong (\mathbb{R}^*)^d$ or $\mathbb{C}^*)^d$).

First we start with the case $K = \mathbb{R}$.

The algebraic torus $\mathbb{T}^d(\mathbb{R})$ has 2^d connected components which are isomorphic as topological groups to $\mathbb{R}^d$. It is enough to understand the topology of $Z_f(\mathbb{R})$ in only one connected component of $\mathbb{T}^d(\mathbb{R})$. In the case $d = 1$ the latter is equivalent to counting positive real roots of a polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{R}[x].$$

The following classical theorem is well-known:

Theorem 7.1. (DÉSCART) *The number of positive real roots of f is not greater than the number of changes of sign in the sequence*

$$\{a_0, a_1, \ldots, a_n\}.$$

In order to get more precise information about positive roots we consider a one-parameter family of polynomials

$$f_t(x) = a_0 t^{m_0} + a_1 x t^{m_1} + \cdots + a_n x^n t^{m_n}$$

for some generic sequence of integers $(m_0, m_1, \ldots, m_n)$. Let $New(f_t)$ be the Newton diagram of $f_t(x)$. Then the low boundary of $New(f_t)$ induces a subdivision of the interval $[0, n]$ into subintervals

$$[i_0, i_1],\ [i_1, i_2],\ \cdots, [i_s, i_{s+1}],$$

where $i_0 = 0$, $i_{s+1} = n$ and $0 < i_1 < i_2 < \cdots < i_s < n$.

Theorem 7.2. *For sufficiently small positive values of t, the number of positive real roots of the polynomial $f_t(x)$ is equal to the number of changes of sign in the sequence*

$$\{a_{i_0}, a_{i_1}, \ldots, a_{i_{s+1}}\}.$$

Proof. The statement follows again from the classical Newton-Puiseux theorem if we remark that the polynomial

$$t^{m_{i_k}} a_{i_k} + t^{m_{i_{k+1}}} a_{i_{k+1}}$$

has a unique positive real root if $a_{i_k} a_{i_{k+1}} < 0$ and has no positive real roots if $a_{i_k} a_{i_{k+1}} > 0$. □

A higher dimensional generalization of the last theorem is due to Viro [58, 47] (see also [28], Chapter 11, §5).

Let A be a finite subset in M such that the convex hull of A is a d-dimensional polytope in $M_{\mathbb{R}}$. Take a convex piecewise linear function φ on Δ which defines a convex triangulation $\mathcal{T}$ of Δ associated with A. For any simplex $\tau \in \mathcal{T}$ of positive dimension which is not contained in the boundary $\partial\Delta$ define a subset $\tau(\mathbb{R})$ as follows. Let $v_1, \ldots, v_s$ be vertices of τ and we assume without loss of generality that

$$a_{v_1}, a_{v_2}, \ldots, a_{v_r} > 0$$

and

$$a_{v_{r+1}}, a_{v_{r+2}}, \ldots, a_{v_s} < 0.$$

Then we define $\tau(\mathbb{R})$ to be the intersection of the relative interior of τ with the affine hyperplane $l(y) = 0$, where $l\ :\ M_{\mathbb{R}} \to \mathbb{R}$ is an arbitrary affine linear function such that

$$l(v_1) = l(v_2) = \cdots = l(v_r) = 1,\ \ l(v_{r+1}) = l(v_{r+2}) = \cdots = l(v_s) = -1.$$

Define

$$\mathcal{T}(\mathbb{R}) = \bigcup_{\tau \not\in \partial\Delta} \tau(\mathbb{R}),$$

where the union runs over all simplices $\tau \in \mathcal{T}$ of positive dimension which are not contained in the boundary $\partial\Delta$.

The theorem of Viro claims:

Theorem 7.3. *For sufficiently small positive values of t, the set of real solutions of $f_t(x)$ in the connected component of unity of $\mathbb{T}^d(\mathbb{R})$ is topologically equivalent to $\mathcal{T}(\mathbb{R})$.*

Example 7.4. Take $A = \{v_0, v_1, \dots, v_d, v_{d+1}\} \subset \mathbb{Z}^d$, where $v_1, \dots, v_d$ is a standard basis of $\mathbb{Z}^d$, $v_0 = 0$ and $v_{d+1} = -v_1 - \cdots - v_d$. Consider a one-parameter family of hypersurfaces

$$f_t(x) = -1 + x_1 + \cdots + x_d + \frac{t}{x_1 \cdots x_d}.$$

Then for sufficiently small positive t, the set of positive real solutions of the equation $f_t(x) = 0$ is topologically equivalent to a $(d-1)$-dimensional unit sphere $y_1^2 + \cdots + y_d^2 = 1$ $(y \in \mathbb{R}^d)$.

The method of Viro can be extended for the study of topology of $Z_f(\mathbb{C})$ using so called **tropical geometry** [40]. We illustrate this method for the case $d = 2$.

Example 7.5. Assume that a 2-dimensional lattice polytope Δ is triangulated into a union of basic triangles $\{\tau_1, \dots, \tau_k\}$, i.e. $\mathrm{Vol}(\tau_i) = 1$. We associate with every basic triangle $\tau_i \in \mathcal{T}$ a 2-dimensional topological manifold $X(\tau_i)$ which is homeomorphic to $S^2 \setminus \{p_1^{(i)}, p_2^{(i)}, p_3^{(i)}\}$ (we establish a bijection between 3 points $p_1^{(i)}, p_2^{(i)}, p_3^{(i)}$ in S^2 and 3 sides of the triangle τ_i). We remark that $X(\tau_i)$ is isomorphic to the algebraic curve in $(\mathbb{C}^*)^2$ defined by the equation $1 + x_1 + x_2 = 0$. It is clear that X_i homeomeorphic to S^2 minus 3 small closed discs $D(p_1^{(i)}), D(p_2^{(i)}), D(p_3^{(i)})$ around the points $p_1^{(i)}, p_2^{(i)}, p_3^{(i)}$. If a 1-dimensional side σ_l of τ_i is not contained in the boundary of Δ then we modify $X(\tau_i)$ by adding $X(\sigma_l) := \partial D(p_l^{(i)}) \cong S^1$ to $X(\tau_i)$ as boundary. By this procedure we obtain a partial compactification $\overline{X(\tau_i)}$ of $X(\tau_i)$ The topological model of a complex Δ-nondegenerate hypersurface $Z_f(\mathbb{C}) \subset \mathbb{T}^2(\mathbb{C})$ is obtained by glueing

$$\overline{X(\tau_1)}, \dots, \overline{X(\tau_k)}$$

along 1-dimensional boundaries $X(\sigma_l)$: if σ_l is a common side of two triangles τ_i and τ_j, then we identify $\overline{X(\tau_i)}$ and $\overline{X(\tau_j)}$ along the circle $X(\sigma_l)$.

Unfortunately, in higher dimensions one can not expect that a d-dimensional lattice polytope Δ admits a convex triangulation $\mathcal{T}$ by basic simplices (such a triangulation is called unimodular). Therefore one needs to consider more general building blocks for topological models of $Z_f(\mathbb{C})$. Consider an arbitrary k-dimensional simplex $\tau \in \mathcal{T}$ $(k > 0,\ \ \tau \not\subset \partial\Delta)$. We denote by $M(\tau)$ the k-dimensional lattice obtained as intersection of M with the minimal affine subspace in $M_{\mathbb{R}}$ containing all vertices $v_0, v_1, \dots, v_k$ of τ. We associate with every such k-dimensional simplex $\tau \in \mathcal{T}$ a $(k+d-2)$-dimensional variety $X(\tau)$ which is isomorphic to the product

$$Z(\tau) \times T(\tau),$$

where $T(\tau) \cong (S^1)^{d-k}$ is a $(d-k)$-dimensional topological torus whose lattice of characters is $M/M(\tau)$ and $Z(\tau) \subset \mathbb{T}_\tau$ affine hypersurface in the algebraic torus $\mathbb{T}_\tau \cong (\mathbb{C}^*)^k$ ($M(\tau)$ is the lattice of algebraic characters of $\mathbb{T}_\tau$) defined by the equation

$$\sum_{i=0}^{k} x^{v_i} = 0.$$

The topological model of a Δ-nondegenerated affine hypersurface $Z_f(\mathbb{C})$ is obtained as disjoint union

$$\amalg_{\tau \not\subset \partial\Delta} X(\tau).$$

This description defines a natural topological filtration $Z_f(\mathbb{C})$ by closed subsets corresponding to union of $X(\tau)$ over all simplices τ of dimension $\leq k$. This filtration induces a filtration $F_{\mathcal{T}}$ in the cohomology $H^{d-1}(Z_f)$.

Theorem 7.6. *Assume that a convex triangulation $\mathcal{T}$ is defined by a piecewise linear convex function $\varphi : \Delta \to \mathbb{R}$ such that $\varphi(A) \subset \mathbb{Z}$ and define*

$$f_t(x) := \sum_{m \in A} a_m t^{\varphi(m)} x^m.$$

Then the filtration $F_{\mathcal{T}}$ in $H^{d-1}(Z_f)$ coincides with the monodromy filtration around $t = 0$ for the 1-parameter family of hypersurfaces defined by the equation $f_t(x)$. If $\mathcal{T}$ is a unimodular conex triangulation then the filtration $F_{\mathcal{T}}$ is opposite to the Hodge filtration in $H^{d-1}(Z_f)$.

Example 7.7. In the previous example 7.5, the dual classes of 1-dimensional cycles $X(\sigma_j)$ generate a $l^*(\Delta)$-dimensional subspace in $H^1(Z_f)$ which is complementary to the Hodge subspace

$$F^1 H^1(Z_f) = H^{1,0} \oplus H^{1,1} \subset H^1(Z_f)$$

of dimension $l^*(\Delta) + B(\Delta) - 1$.

8. Toric mirror symmetry

We start with one version of toric mirror symmetry which is called **local mirror symmetry**.

Let $A = \{v_1, \dots, v_n\}$ be a finite subset in M such that $\langle v_i, u\rangle = 1$ $(i = 1, \dots, n)$ for some lattice point $u \in N$. We assume that the convex hull of A is a $(d-1)$-dimensional polytope Δ. Then we have a short exact sequence

$$0 \to R(A)_{\mathbb{R}} \to \mathbb{R}^n \to M_{\mathbb{R}} \to 0.$$

Consider the dual exact sequence

$$0 \to N_{\mathbb{R}} \to \mathbb{R}^n \xrightarrow{\alpha} R(A)^*_{\mathbb{R}} \to 0.$$

Then the secondary polytope $\mathrm{Sec}(A)$ as a polytope in $R(A)_{\mathbb{R}}$ defines a secondary fan $\Sigma(A)$ in the dual space $R(A)^*_{\mathbb{R}}$. A cone σ of maximal dimension in $\Sigma(A)$ is the set of all linear functions on $R(A)_{\mathbb{R}}$ which attain minimum at the same vertex of the secondary polytope $\mathrm{Sec}(A)$.

Let $\mathbb{C}^n \to \mathbb{R}^n$ be the smooth map

$$(z_1, \dots, z_n) \mapsto (|z_1|^2, \dots, |z_n|^2).$$

Its composition with the linear map $\alpha : \mathbb{R}^n \to R(A)^*_{\mathbb{R}}$ will be denoted by μ_A.

Theorem 8.1. *Let p be an arbitrary point in the $(n-d)$-dimensional space $R(A)^*_{\mathbb{R}}$. The subset $\mu_A^{-1}(p) \subset \mathbb{C}^n$ is a smooth real subvariety if and only if p is an interior point of a $(n-d)$-dimensional cone $\sigma \in \Sigma(A)$. If we consider*

$$H(z) = \sum_{i=1}^n |z_i|^2$$

as a Hamilton function on $\mathbb{C}^n$, then

$$\mu_A : \mathbb{C}^n \to R(A)^*_{\mathbb{R}}$$

is the moment mapping corresponding to a Hamiltonian action of the $(n-d)$-dimensional compact torus $T(A) \subset (S^1)^n$ whose lattice of characters is dual to $R(A)$. In particular,

$R(A)^*_{\mathbb{R}}$ can be identified with the dual space to Lie algebra of $T(A)$. If $\mu_A^{-1}(p)$ is smooth, then the torus action of $T(A)$ on $\mu_A^{-1}(p)$ has at most finite stabilizer and the quotient space

$$X(p) := \mu_A^{-1}(p)/T(A)$$

is a symplectic variety with at worst orbifold singularities. The variety $X(p)$ is smooth if and only if p is an interior lattice point of a $(n-d)$-dimensional cone $\sigma \in \Sigma(A)$ corresponding to a convex unimodular triangulation of Δ.

Let $\psi_0(\Delta), \psi_1(\Delta), \dots, \psi_{d-1}(\Delta)$ (resp. $\varphi_1(\Delta), \varphi_2(\Delta), \dots, \varphi_d(\Delta)$) be the coefficients of the polynomial $\Psi_\Delta(t)$ (resp. $\Phi_\Delta(t)$). Recall that

$$t^d \Psi_\Delta(t^{-1}) = \Phi_\Delta(t).$$

A starting point of local toric mirror symmetry could be the following important observation:

Theorem 8.2. *Assume that $X(p)$ is smooth, then $X(p)$ is a quasi-projective smooth toric variety of dimension d over $\mathbb{C}$ and its cohomology groups $H^i(X(p),\mathbb{Z})$ and $H^i_c(X(p),\mathbb{Z})$ are free abelian groups of whose rank is defined by the following formulas*

$$H^i(X(p),\mathbb{Z}) = H^i_c(X(p),\mathbb{Z}) = 0, \ \text{ if } i \neq 2k,$$

$$\operatorname{rk} H^{2k}(X(p),\mathbb{Z}) = \psi_k(\Delta), \ \ k = 0, 1, \dots, d-1,$$

$$\operatorname{rk} H^{2k}_c(X(p),\mathbb{Z}) = \varphi_k(\Delta), \ \ k = 1, 2, \dots, d.$$

Example 8.3. Let $A = \{(1,0), \dots, (1,n)\} \subset \mathbb{Z}^2$. Then the polytope $\Delta = \operatorname{Conv}(A)$ admits a single unimodular triangulation. The corresponding smooth variety $X(p)$ is a minimal desingularization of the cyclic quotient singularity $x_0^n - x_1x_2 = 0$.

Theorem 8.2 suggests that there should be some relation between the pair of cohomology groups $(H^*(X(p),\mathbb{Z}), H^*_c(X(p),\mathbb{Z}))$ and the pair (S_f, I_f) for Δ-nodegenerate Laurent polynomials f. The second pair depends on the coefficients of f, but the first one has no parameters. One can make the pair $(H^*(X(p),\mathbb{Z}), H^*_c(X(p),\mathbb{Z}))$ to be dependent on parameters corresponding to the choice of the point p. The choice of p determines the symplectic structure on $X(p)$. Using Gromov-Witten invariants of maps of pseudo-holomorphic curves to $X(p)$, one can define a deformation of the pair $(H^*(X(p),\mathbb{Z}), H^*_c(X(p),\mathbb{Z}))$ parametrized by the symplectic structure. Such a deformation is called **(small) quantum cohomology** of $X(p)$.

In physical literature the last construction is called **gauged linear sigma-model** associated with $X(p)$. On the other hand, a Δ-nondegenerate Laurent polynomial f determines so called **Landau-Ginzburg theory** which depends on coefficients of f (only $(n-d)$ of these coefficients can be considered as independent).

In fact, there exist a deformation of pairs $(H^*(X(p),\mathbb{Z}), H^*_c(X(p),\mathbb{Z}))$ and (S_f, I_f) which depend on $\operatorname{Vol}(\Delta)$ independent parameters. For the first pair such a deformation is determined by a **big quantum cohomology** of $X(p)$. For the second pair, the corresponding deformation defined by Barannikov-Kontsevich is determined by the Lie algebra of polyvector fileds. Both deformations can be formalized in the language of Frobenius manifolds [34]. The identification of two Frobenius structures coming from f and $X(p)$ is a very nontrivial problem. This identification is called **toric mirror symmetry**. The first step in this identification is so called **monomial-divisor mirror correspondence** [6]: the set A parametrize simultaneously the set of monomials $a_i x^{v_i}$ $(i = 1, \dots, n)$ in the polynomial f and the set of divisors $z_i = 0$ $(i = 1, \dots, n)$ on $X(p)$.

Originally, toric mirror symmetry was formulated for Calabi-Yau hypersurfaces in toric varieties $\mathbb{P}_\Delta$ corresponding to reflexive polytopes [10]. It was observed in [11] that periods of such hypersurfaces are GKZ-hypergeometric functions.

Definition 8.4. A d-dimensional lattice polytope $\Delta \subset M_{\mathbb{R}}$ is called **reflexive** if 0 is contained in the interior of Δ and the polytope

$$\Delta^* := \{y \in N_{\mathbb{R}} \ : \ \langle x, y\rangle \geq -1 \ \ \forall x \in \Delta\}$$

has vertices in N.

If Δ is reflexive, then Δ^* is also reflexive and $(\Delta^*)^* = \Delta$. Both toric varieties $\mathbb{P}_\Delta$ and $\mathbb{P}_{\Delta^*}$ are Fano varieties with at worst Gorenstein singularities. The combinatorial duality between Δ and Δ^* is a basis of toric mirror symmetry for Calabi-Yau hypersurfaces in Gerenstein toric Fano varieties [10].

Example 8.5. Let $v_1, \dots, v_d$ be a standard basis of the lattice Z^d. Define Δ to be the convex hull of $v_1, \dots, v_d$ and $v_{d+1} = -v_1 - \cdots - v_d$. Then Δ is a simplest example of a d-dimensionale polytope. The dual reflexive polytope is the simplex

$$\Delta^* = \{(y_1, \dots, y_d) \ : \ y_1, \dots, y_d \geq -1, \ y_1 + \cdots + y_d \leq 1.\}.$$

These two reflexive polytopes define families of Calabi-Yau hypersurfaces of degree $(d+1)$ in $\mathbb{P}^d$ and their mirrors.

If Δ is a d-dimensional reflexive polytope then I_f is isomorphic to $S_f(-1)$ as homogeneous S_f-module. In particular, we have

$$\psi_i(\Delta) = \varphi_{i+1}(\Delta), \ \ i = 0, 1, \dots, d.$$

The ring S_f can be considered as a deformation of the (orbifold) cohomology ring of the toric Fano variety $\mathbb{P}_{\Delta^*}$.

Mirror symmetry for Calabi-Yau manifolds exchange the deformation of complex structure and with the deformation of symplectic structure. In particular, both deformations are described by regular holonomic differential systems.

The Gromov-Witten deformations in the quantum cohomology of Fano varieties are expected to be descibed by holonomic differential systems corresponding to oscillating integrals. Singularities of these differential systems are not regular anymore.

Example 8.6. Let $A = \{-1, 1\}$. Consider the integral

$$\frac{1}{2\pi i}\int_{|z|=1} \exp\left(z + \frac{t}{z}\right)\frac{dz}{z} = \sum_{k\geq 0}\frac{1}{k!}\frac{1}{2\pi i}\int_{|z|=1}\left(z+\frac{t}{z}\right)^k \frac{dz}{z} =$$

$$= \sum_{l \geq 0} \frac{1}{(2l)!}\binom{2l}{l} t^l = \sum_{l\geq 0}\frac{t^l}{(l!)^2}.$$

This series is known to be the solution of the differential equation

$$\Theta^2 \Phi(t) = t\Phi(t), \ \ \Theta := t\frac{\partial}{\partial t}$$

corresponding to the small quantum cohomology of $\mathbb{P}^1$ [30]. This equation has irregular singularities.

Remark 8.7. As we have already mentioned, Gromov-Witten invariants obtained from intersection theory for maps of genus-0 pseudo-holomorphic curves to a given compact symplectic manifold X define a structure of Frobenius manifold on the cohomology of X. However, it is not much known about the structure behind the Gromov-Witten invariants obtained from intersection theory for maps of higher genus pseudo-holomorphic curves to X. Only the following two cases have been investigated in details:

Case 1: $A = \{-1, 1\}$. The higher genus Gromov-Witten theory on $\mathbb{P}^1$ has an elegant description in terms of integrable systems (Toda-hierarchy). This case was investigated recently by Getzler, Okounkov and Pandharipande [46, 44, 29]

Case 2: $A = \{(1,0), \ldots, (1,n)\} \subset \mathbb{Z}^2$. It was shown by Givental [31] that the total descendent Gromov-Witten potential for mthe inimal symplectic resolution of A_{n-1}-singularity is the Witten-Kontsevich τ-function of the integrable nKdV-hierarchy. Similar results hold for minimal resolutions of arbitrary ADE-singularities [32].

9. Some student projects

1. Is it possible to generalize the notions of principal A-determinant and secondary polytope for nonregular holonomic (confluent) A-hypergeometric systems? One would like to find different power series expansions of generalized hypergeometric functions parametrized by "vertices of the secondary polytope" as in the regular case.

2. Find a combinatorial formula for all twisted Hodge-Deligne numbers of toric hypersurfaces Z_f. The complete combinatorial formulas in the nontwisted case were obtained [12]. Does there exist a twisted version of toric mirror duality.

3. One could use the classical formulas for Γ-function

$$\Gamma(\beta) = \int_0^\infty z^\beta \exp(-z) \frac{dz}{z},$$

$$\frac{\Gamma(\beta)}{a^\beta} = \int_0^\infty z^\beta \exp(-az) \frac{dz}{z}$$

and define a formal solution to GKZ-hypergeometric system in the form

$$\Phi^*_\lambda = \int_{iR(A)_\mathbb{R}} \left(\int_{\mathbb{R}^n_{>0}} z^{(x+\lambda)} \exp\left(-\sum_{i=1}^n a_i z_i \right) \frac{dz}{z} \right) dx =$$

$$= \int_{iR(A)_\mathbb{R}} a^{-x-\lambda} \prod_{i=1}^n \Gamma(x_i + \lambda_i) dx.$$

This integral is analogous to Mellin-Barnes integral for classical hyprgeometric functions. Is it possible to compute these integrals using multidimensional residues (corresponding to poles of Γ-functions) and write down them as as explicit powers series?

4. Let $Z_f \subset \mathbb{T}^d$ be a Δ-nondegenerate hypersurface. Consider a convex triangulation $\mathcal{T}$ of Δ corresponding to a vertex of the secondary polytope. Find explicit power series expansion of periods of holomorphic forms on Z_f on $(d-1)$-dimensional cycles represented by connected components of $Z_f(\mathbb{R})$ (real solutions of the equation $f = 0$. For example, one knows that

the integral of the holomorphic 3-form on a Calabi-Yau quintic 3-fold V over p-adic points $V(\mathbb{Q}_p)$ is connected to counting of rational points modulo p and to the hypergeometric series

$$\sum_{n\geq 0} \frac{5n!}{(n!)^5} z^n.$$

Is this series related to the integral of the holomorphic 3-form over $V(\mathbb{R})$? May be it would be easier to start with the case of elliptic curves [13].

5. Find confirmations of the toric (local) mirror symmetry for Gorenstein toric singularties from view-point of monodromy groups. The case of 2-dimensional Gorenstein toric singularities was considered by Skarke [52].

6. Find confirmations of the toric (global) mirror symmetry for new examples of toric Fano varieties. The main motivation is the paper of Barannikov [7] about mirror of $\mathbb{P}^n$. The corresponding oscillating integrals are analogous to hyper-Kloosterman sums [53]. Similar ideas related to Frobenius manifolds are contained in the recent papers of Douai and Sabbah [18, 19]

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