Periods and Zeta Functions

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Course Description

The lectures will be concerned with matters relating to the calculation of the numbers of points of Calabi–Yau manifolds over the finite fields $\mathbb{F}_q$ with q a power of a prime p. We shall be guided by a desire to perform explicit calculations within the context of toric geometry. Given computable expressions for the number of points it is then natural to consider the form of the zeta function for these manifolds. In particular it is natural to examine the relation of mirror symmetry to the form of the zeta function.

Two families of threefolds will figure prominently in our discussion. The first of these is a one parameter family of quintic hypersurfaces in $\mathbb{P}^4$ given by the equation

$$P(x, \psi) \;=\; \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5 \;.$$

The second example is a two parameter family of octic hypersurfaces in the weighted projective space $\mathbb{P}^4_{(1,1,2,2,2)}$ [that is the set of points $(x_1, x_2, x_3, x_4, x_5)$, whose coordinates are not all zero and are subject to the identification $(x_1, x_2, x_3, x_4, x_5) \sim (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^2 x_4, \lambda^2 x_5)$ for all nonzero λ].

$$P(x, \psi, \phi) \;=\; x_1^8 + x_2^8 + x_3^4 + x_4^4 + x_5^4 - 8\psi\, x_1 x_2 x_3 x_4 x_5 - 2\phi\, x_1^4 x_2^4 \;.$$

These families of manifolds are chosen because they are among the simplest Calabi–Yau threefolds, they have been well studied over $\mathbb{C}$ and their mirrors are easily described.

Course Content

- The relation of the number of $\mathbb{F}_q$-rational points to the periods of the manifold.
- Derivation of the related expressions for the number of points in terms of Gauss sums.
- The discriminant locus of the moduli space of the manifolds.
- The form of the zeta function.
- Mirror symmetry and the large complex structure limit.

Student Project

It is hard to perform explicit calculations without dedicating much time to the writing of computer codes. The following are interesting and relevant problems which can be investigated with the aid of Mathematica or an equivalent package. I have not performed any prior investigation of these problems so do not know if they are easy or if they are Ph.D. projects.

1) For the conifold points of the quintic, for which $\psi^5 = 1$ the zeta function consists of trivial factors together with the notrivial factor $(1 - a_p t + p^3 t^2)$, where it is known that a_p is the p'th coefficient in the q-expansion of the unique cusp form, f, of weight 4 for the group $\Gamma_0(25)$

$$\begin{aligned} f(q) &= \eta(q^5)^4 \left[\eta(q)^4 + 5\,\eta(q)^3\eta(q^{25}) + 20\,\eta(q)^2\eta(q^{25})^2 + 25\,\eta(q)\eta(q^{25})^3 + 25\,\eta(q^{25})^4\right] \\ &= q + q^2 + 7q^3 - 7q^4 + 7q^6 + 6q^7 - 15q^8 + 22q^9 - 43q^{11} - 49q^{12} - 28q^{13} + 6q^{14} \\ &\quad + 41q^{16} + 91q^{17} + 22q^{18} - 35q^{19} + 42q^{21} - 43q^{22} + 162q^{23} - 105q^{24} - 28q^{26} - 35q^{27} \\ &\quad - 42q^{28} + 160q^{29} + 42q^{31} + 161q^{32} - 301q^{33} + 91q^{34} - 154q^{36} - 314q^{37} - 35q^{38} \\ &\quad - 196q^{39} - 203q^{41} + 42q^{42} + 92q^{43} + 301q^{44} + 162q^{46} + 196q^{47} + 287q^{48} + \cdots \ . \end{aligned}$$

Thus $a_3 = 7$ and $a_7 = 6$ etc. and in the above η denotes the Dedekind function

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \ .$$

Interest attaches to the 5-adic expansion of the coefficients a_p in relation to mirror symmetry and the large complex structure limit. For example, at least for the first 500 primes that I have checked, we have

$$a_p \ \equiv \ p + p^2 \bmod 5 \quad \text{and} \quad a_p \ \equiv \ p + p^2 \bmod 5^2 \quad \text{if} \ \ 5|(p-1) \qquad (*)$$

so that $(1 - a_p t + p^3 t^2) \equiv (1 - pt)(1 - p^2 t)$.

The project then is to prove (*) for general p and/or see if something interesting can be said about the higher terms in the 5-adic expansion of the coefficients a_p.

2) For an elliptic curve corresponding to the equation $x_1^3 + x_2^3 + x_3^3 - 3\phi x_1 x_2 x_3 = 0$ there is an analogous factor $(1 - b_p(\phi)t + pt^2)$ in the zeta function. Is it possible to say something interesting about the 3-adic expansion of the coefficients $b_p(\phi)$?

Background Reading

Some familiarity with material to be covered by the other lecture courses at the School viz. toric geometry, periods, mirror symmetry and p-adic analysis will be highly desirable and the student is referred to the background reading lists of the relevant courses for these topics. The following are additional references close in style to the course:

P. Candelas, X. de la Ossa and F. Rodriguez-Villegas,
Calabi–Yau manifolds over Finite Fields I, 76pp, hep-th/0012233.

P. Candelas, X. de la Ossa and F. Rodriguez-Villegas,
Calabi–Yau manifolds over Finite Fields II, 50pp, hep-th/0402133.

P. Candelas, X. de la Ossa, A. Font, S. Katz, D. R. Morrison,
Mirror Symmetry for Two Parameter Models I
Nucl. Phys. **B416** 481-538, 1994, hep-th/9308083. (See especially §3).