Math 124 - Exam 3 Practice problems- Spring '06

1. A window frame has the shape of a rectangle with a semicircle on top. All four sides of the rectangle are part of the frame. The straight portions of the frame (the four sides of the rectrangle) cost 8$ per foot, and the curved portion (the semicircle) costs 12$ per foot. The total area of the window must be 20 square feet. Minimize the cost of the frame.

Let x be the width of the frame, y the height of the rectangular part. So the radius of the semicircle is $x/2$. The cost is

$$C = 8(2y + 2x) + 12(\pi x/2) = 16y + 16x + 6\pi x$$

The area must be 20, so

$$20 = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = xy + \frac{1}{8}\pi x^2$$

Solve this for y:

$$y = \frac{20 - \frac{1}{8}\pi x^2}{x}$$

Then plug this in for y in the formula for C:

$$\begin{aligned} C &= 16\frac{20 - \frac{1}{8}\pi x^2}{x} + 16x + 6\pi x \\ &= \frac{320}{x} - 2\pi x + 16x + 6\pi x = \frac{320}{x} + 16x + 4\pi x \end{aligned} \tag{1}$$

Set $dC/dx = 0$

$$0 = \frac{-320}{x^2} + 16 + 4\pi \tag{2}$$

So

$$x = \sqrt{\frac{320}{16 + 4\pi}} \approx 3.347 \tag{3}$$

This gives $y \approx 4.661$. The cost is $C = 16y + 16x + 6\pi x = \$191.22$

2. $f(x) = xe^{-x^2}$
(a) Find all x values at which f has a local min or max.
f has a local min at $-1/\sqrt{2}$ and a local max at $1/\sqrt{2}$.
(b) Find all inflection points.
f has inflection points at $x = 0, \sqrt{3/2}, -\sqrt{3/2}$.
(c) Find the global min and max over $0 \leq x \leq 1$.
The only critical point inside this interval is $1/\sqrt{2}$. So the global min and max will occur at $x = 0, 1/\sqrt{2}$ or 1. $f(0) = 0$, $f(1/\sqrt{2}) = e^{-1/2}/\sqrt{2} \approx 0.4288$ and $f(1) = e^{-1} \approx 0.3678$. So the global min of f is 0 and is attained at $x = 0$ and the global max of f is $e^{-1/2}/\sqrt{2} \approx 0.4288$, and is attained at $x = 1/\sqrt{2}$.

3.

$$\lim_{x\to\infty} \frac{1-\cos(ax)}{x^2},$$

This is NOT a L'Hopital problem. The numerator is bounded between 0 and 2 and the denominator goes to ∞, so the fraction converges to 0.

$$\lim_{x\to 0} \frac{1-\cos(ax)}{x^2},$$

The numerator and denominator both go to 0, so we can apply L'Hopital. In fact, you have to apply it twice :

$$= \lim_{x\to 0} \frac{a\sin(ax)}{2x} = \lim_{x\to 0} \frac{a^2\cos(ax)}{2} = \frac{a^2}{2}$$

$$\lim_{x\to\infty} \frac{\ln(x)}{\sinh(x)}$$

The numerator and denominator both go to ∞, so we apply L'Hopital:

$$= \lim_{x\to\infty} \frac{1/x}{\cosh(x)}$$

Now the numerator goes to zero and the denominator goes to ∞. So the fraction goes to 0.

4. Let $f(x) = 5a^3x^2 - 2x^5$. Here a is a parameter; it does not depend on x.
(a) Find the critical points and determine if they are local min or maxs. Your answer should involve a.

$$f'(x) = 10a^3x - 10x^4 = 10x(a^3 - x^3)$$

So we have critical points at $x = 0$ and $x = a$.

First consider the case of $a > 0$. For $x < 0$ it is easy to see $f'(x) < 0$. And when $x \to \infty$, $f'(x)$ is negative. For $0 < x < a$, $a^3 - x^3$ will be positive, so $f'(x)$ will be positive. So by the first derivative test, $x = 0$ is a local min, $x = a$ is a local max.

Now consider the case of $a < 0$. So the critical point at a is now left of the one at 0. For $x > 0$, $a^3 - x^3$ will be negative, so $f'(x)$ will be negative. when $x \to \infty$, $f'(x)$ is negative. So $x = a$ will be a local min and $x = 0$ will be a local max.

Finally consider the case that $a = 0$. Then $f(x) = -2x^5$. This has one critical point at $x = 0$ and it is neither a local min or max.
(b) Find the global max over $x \geq 0$.

First note that as $x \to \infty$, $f(x)$ goes to $-\infty$, regardless of the value of a. So this will never be the global max. If $a > 0$ we have to compare $f(0) = 0$ and $f(a) = 3a^5$. So the global max is $3a^5$ and it is attained at $x = a$.

If $a < 0$ we need only consider $x = 0$. $f(0) = 0$, so the global max is 0 and it is attained at $x = 0$.

If $a = 0$ the global max is 0 and it is attained at $x = 0$.

To check your answers, graph the function for two values of a, one positive and one negative.

5. Consider the curve $x^3 + y^2\cosh(y - 1) = 2$
(a) Find the equation of the tangent line at the point $(1, 1)$ to the curve .

Implicit differentiation:

$$3x^2 + 2y\cosh(y-1)\frac{dy}{dx} + y^2\sinh(y-1)\frac{dy}{dx} = 0$$

Plug in $x = 1$ and $y = 1$,

$$3 + 2\cosh(0)\frac{dy}{dx} + \sinh(0)\frac{dy}{dx} = 0$$

Using $\cosh(0) = 1, \sinh(0) = 0$,

$$3 + 2\frac{dy}{dx} = 0$$

So $\frac{dy}{dx} = -3/2$. The tangent line must go through $(1, 1)$, so its equation is $y - 1 = -3/2(x - 1)$, or $y = -3x/2 + 5/2$.

(b) Use your answer to (a) to find approximately the value of y so that $(1.01, y)$ is also on the curve.

Use the tangent line to approximate the curve: $y = -1.5 \times 1.01 + 5/2 = 0.985$.