Math 129 (Fall '05) - Exam 3 Solutions - Kennedy

1. (16 points) Consider the differential equation and initial condition.

$$\frac{dy}{dx} = -x(y-1), \quad y(0) = 2 \tag{1}$$

(a) Use Euler's method with $\Delta x = 0.1$ to find $y(1)$ approximately.

Answer: Calculator program should give 1.628156 .

(b) Find the exact solution of the differential equation.

Answer: Use separation of variables:

$$\frac{dy}{y-1} = -x\,dx \tag{2}$$

Integrate to get

$$\ln|y-1| = -\frac{1}{2}x^2 + C \tag{3}$$

To get $y(0) = 2$ we must have $C = 0$. So

$$\ln|y-1| = -\frac{1}{2}x^2 \tag{4}$$

Exponentiate:

$$|y-1| = e^{-x^2/2} \tag{5}$$

To get $y(0) = 2$ we must take $y - 1 > 0$, so

$$y = e^{-x^2/2} + 1 \tag{6}$$

(c) Find the limit of the solution as $x \to \infty$. You can do this part without having done any of the previous parts.

Answer: Using the answer to (b) we see the limit is 1. You can also see this from the slope field and you could even use your Euler program to find the limit.

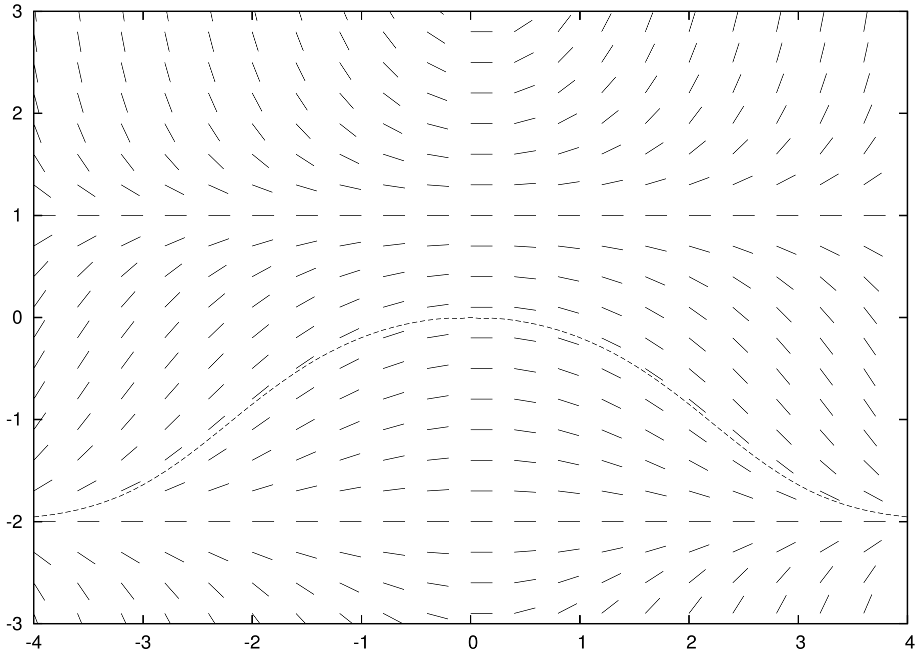
2. (12 points) The slope field for $\frac{dy}{dx} = x(y+2)(y-1)/5$ is shown below.

(a) Sketch the solution which goes through the origin. (Your sketch should include the solution for negative values of x.)

Answer: See the graph.

(b) For what constants c does the function $y(x) = c$ solve this differential equation?

Answer: If $y(x)$ is a constant, then $\frac{dy}{dx}$ is zero. So we get a constant solution only when $x(y+2)(y-1)/5$ is zero for all x. This will happen only if $y = -2$ or $y = 1$.



3. (10 points) For the differential equation $\frac{d^2y}{dx^2} - 4y = 0$ one of the functions e^{kx} and $\sin(kx)$ is a solution for some values of k. Determine which function is a solution and for what values of k it is a solution.

Answer: For $y(x) = e^{kx}$, $\frac{d^2y}{dx^2} - 4y = (k^2 - 4)e^{kx}$. This is zero if $k^2 = 4$, i.e., if $k = 2$ or -2. So e^{kx} is a solution for these two values of k.

For $y(x) = \sin(kx)$, $\frac{d^2y}{dx^2} - 4y = (-k^2 - 4)\sin(kx)$. This cannot be zero for all x no matter what value of k we try. So $\sin(kx)$ is never a solution.

4. (14 points) Find the radius of convergence of the following power series.

$$(x-1) + 2(x-1)^2 + 3(x-1)^3 + 4(x-1)^4 + 5(x-1)^5 + 6(x-1)^6 + \cdots$$

Answer: As always, we use the ratio test to do this. The general term in the above is $n(x-1)^n$. So the ratio is

$$\frac{(n+1)|x-1|^{n+1}}{n|x-1|^n} = \frac{n+1}{n}|x-1| \tag{7}$$

As $n \to \infty$, $(n+1)/n$ converges to 1, so the above converges to $|x-1|$. Thus the series converges if $|x-1| < 1$, i.e., for $0 < x < 2$. So the radius of convergence is 1.

5. (14 points) (a) Find the Taylor series of xe^{x^2} about 0. Your answer should either give the general term or include enough terms to make the pattern clear.

Answer:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{8}$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \tag{9}$$

$$xe^{x^2} = x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!} \tag{10}$$

(b) What is the fifth derivative of $2xe^{x^2}$ at $x = 0$?

Answer: In the Taylor series the coefficient of the x^5 term is $f^{(5)}(0)/5!$. From the above we see that this coefficient is $1/2! = 1/2$ for xe^{x^2} and so is 1 for $2xe^{x^2}$. So $f^{(5)}(0)/5! = 1$, which implies $f^{(5)}(0) = 5! = 120$.

6. (21 points) For each of the following series state whether it converges or not and give the name of the test that gives your answer. In the case of the comparison test you should say what series you are comparing with.

$$\sum_{n=0}^{\infty} \frac{1}{(n+5)^2} \tag{11}$$

Answer: This can be done with either the comparison test or the integral test. For the comparison test, we have

$$\frac{1}{(n+5)^2} \leq \frac{1}{n^2} \tag{12}$$

We know $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$, so the comparison test says the series converges.

For the integral test you compute

$$\int_1^\infty \frac{1}{(x+5)^2}\,dx \tag{13}$$

and see that it converges, so the series does too.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(-2)^n} \tag{14}$$

Answer: Use the ratio test. Ratio is (remember the absolute values)

$$\frac{\sqrt{n+1}}{2^{n+1}}\frac{2^n}{\sqrt{n}} = \frac{\sqrt{n+1}}{\sqrt{n}}\frac{1}{2} \tag{15}$$

This converges to 1/2, so the ratio test says the series converges.

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{3^n} \tag{16}$$

Answer: Since $\cos^2(n) \leq 1$ we have

$$\frac{\cos^2(n)}{3^n} \leq \frac{1}{3^n} \tag{17}$$

We know that

$$\sum_{n=1}^{\infty} \frac{1}{3^n} \tag{18}$$

conveges since it is a geometric series with $r = 1/3 < 1$. So the comparison test says the original series converges.

7. (13 points) Let $f(x) = \sin(x)$.
(a) Find the third order Taylor polynomial for $f(x)$ about $x = \pi/4$. **Note** that it is not about $x = 0$.

Answer:

$$\begin{aligned} f(x) &= \sin(x), \quad f(\pi/4) = 1/\sqrt{2} \\ f'(x) &= \cos(x), \quad f'(\pi/4) = 1/\sqrt{2} \\ f^{(2)}(x) &= -\sin(x), \quad f^{(2)}(\pi/4) = -1/\sqrt{2} \\ f^{(3)}(x) &= -\cos(x), \quad f^{(3)}(\pi/4) = -1/\sqrt{2} \end{aligned} \tag{19}$$

So the Taylor polynomial is

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \pi/4) - \frac{1}{2\sqrt{2}}(x - \pi/4)^2 - \frac{1}{6\sqrt{2}}(x - \pi/4)^3 \tag{20}$$

(b) Find a bound on the error if we use this Taylor polynomial to approximate $\sin(\pi/4 + 0.1)$.

Answer: We are using $a = \pi/4$ and $x = \pi/4 + 0.1$ and $n = 3$. $|f^{(4)}(x)| = \sin(x)$. The max of this between a and x occurs at $x = \pi/4 + 0.1$. So

$$M = \sin(\pi/4 + 0.1) \tag{21}$$

Plugging into the error formula

$$error \leq \frac{\sin(\pi/4 + 0.1)}{4!}(0.1)^4 = 3.22 \times 10^{-6} \tag{22}$$