Math 129 (Fall '05) - Sample Exam 2 - Kennedy

NOTE: These solutions just give some of the steps in the solution. On the exam be sure to sure all of your work.

WARNING: This is a past exam I gave. You should not assume that if a certain topic does not appear here it will not appear on our exam.

1. (14 points) Consider the area enclosed by the y-axis, the x-axis, the vertical line $x = 1$ and the curve $y = e^{-x}$. It is rotated about the y-axis. Find the volume of the resulting solid. For full credit you must do the integral analytically. However, a numerical answer is better than nothing.

If you slice it vertically the slices give cylindrical shells. The total volume is

$$\int_0^1 2\pi x e^{-x}\, dx$$

Integration by parts or the tables show this equals

$$= \left[-xe^{-x} - e^{-x}\right]_0^1 = 2\pi(1 - \frac{2}{e})$$

If you slice it horizontally the slices are discs. The radius of the disc is 1 if $y \leq 1/e$. For $y > 1/e$ the radius is $x = \ln y$. So the total volume is

$$\int_{1/e}^1 \pi(\ln y)^2\, dy + \int_0^{1/e} \pi 1^2\, dy$$

The first integral is done by parts.

$$\int (\ln y)^2\, dy = y(\ln y)^2 - 2\int \ln y\, dy = y(\ln y)^2 - 2y\ln y + 2y$$

and this leads to the same answer as the above.

2. (14 points) A water truck weighs 10,000 lbs when it is full of water. The truck starts up a mountain road full of water. The truck travels at a constant speed and the road has a constant incline. At the start of the trip the truck springs a leak. Water leaks out at a constant rate and at the top the truck only weighs 6,000 lbs. The top of the road is 5,000 feet higher than the bottom.

(a) $W(0) = 10,000$ and $W(5000) = 6,000$. Since the truck is going at a constant speed and the water is leaking at a constant rate, the weight will be a linear function of h. Using the above two points, the slope is $-4000/5000 = -4/5$. So

$$W(h) = 10,000 - \frac{4}{5}h$$

(b) Find the total work done by the truck. When the truck goes up a height Δh the work done is $W(h)\Delta h$. So the total work is

$$\int_0^{5000} W(h)\,dh = \int_0^{5000} (10,000 - \frac{4}{5}h)\,dh = 4 \cdot 10^7 ft - lbs$$

3. (8 points) Find the arc length of the graph of $y = 2x^{3/2}$ for $0 \le x \le 3$.

$$\int_0^4 \sqrt{1+(y')^2} = \int_0^4 \sqrt{1+9x} = \left[\frac{2}{27}(1+9x)^{3/2}\right]_0^4 = \frac{2}{27}((28)^{3/2} - 1)$$

4. (14 points) A dam is 200 feet across the top and 100 feet tall at its midpoint. Its shape is approximately given by the parabola $y = x^2/100$ with $-100 \le x \le 100$. The water behind the dam goes up to the very top of the dam. Find the total force on the dam. (Recall that at a depth of h feet below the surface the water pressure is $62.4h$ lbs per ft^2

Slice the dam horizontally. We take y to be the height from the bottom of the dam. The width of a slice is $2x = 2\sqrt{100y}$. So the area of the slice is $2\sqrt{100y}\Delta y$. The depth of the slice below the surface is $100 - y$. So the pressure on the slice is $62.4(100 - y)$. So the force on the slice is $62.4(100 - y)2\sqrt{100y}\Delta y$. So the total force on the dam is

$$\int_0^{100} 62.4(100-y)2\sqrt{100y}\,dy = 1248 \int_0^{100} (100-y)\sqrt{y}\,dy = 1248 \int_0^{100} (100y^{1/2} - y^{3/2})dy$$

$$= 1248 \left[100\frac{2}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^{100} = 3.327 \cdot 10^7 lbs$$

5. (14 points) A cylindrical barrel is 5 ft tall has a radius of 1.5 ft. It it filled to a depth of 4 ft with a mysterious liquid whose density depends on the depth in the liquid. The density d depends on the distance h below the surface according to $d(h) = 40(1 + h/10)$. (The density is in lbs/ft^3 and the distance h is in feet.) Find the total work needed to pump the liquid to the top rim of the barrel.

Slice it horizontally so the slices are discs. The volume of a disc is $\pi(1.5)^2\Delta h$. The weight of a slice is $d(h)\,\pi(1.5)^2\Delta h$. Note that h is measured from the surface so the slice must be raised a distance of $1 + h$. So the work for the slice is $(1 + h)d(h)\,\pi(1.5)^2\Delta h$. So the total work is

$$\int_0^4 (1+h)d(h)\,\pi(1.5)^2\,dh = \int_0^4 (1+h)40(1+h/10)\pi(1.5)^2\,dh$$

$$= 40\pi(1.5)^2 \int_0^4 (1+h)(1+h/10)\,dh$$

$$= 90\pi \int_0^4 (1 + \frac{11}{10}h + \frac{h^2}{10})dh = 90\pi \left[h + \frac{11}{20}h^2 + \frac{h^3}{30}\right]_0^4$$
$$= 90\pi \left[4 + \frac{11}{20}4^2 + \frac{4^3}{30}\right] = 4222.3ft - lbs$$

6.

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \cdots \frac{1}{2^{20}} = \frac{1 - \frac{1}{2^{21}}}{1 - 1/2} = 2 - \frac{1}{2^{20}}$$

$$-3 + 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} \cdots = -3(1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} \cdots) = -3\frac{1}{1 + 1/3} = \frac{-9}{4}$$

$$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1 - x^2}$$

7. (14 points) A tetrahedron has vertices at $(0, 0, 0)$, $(2, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Find its volume. (A tetrahedron has four faces, each of which is a triangle. Moreover, any slice through a tetrahedron is a triangle.)

Slice it perpendicular to the x-axis. Then the slices are 45-45-90 triangles. Similar triangles show that the short sides of the slice are $1 - x/2$. So the volume of the slice is $1/2(1 - x/2)^2 \Delta x$. So the total volume is

$$\int_0^2 \frac{1}{2}(1 - x/2)^2 \, dx = \int_0^2 \frac{1}{2}(1 - x + x^2/4) \, dx = \left[x/2 - x^2/4 + x^3/24\right]_0^2 = 1/3$$

8. (10 points) Determine whether the following improper integral converges and explain your reasoning.

$$\int_0^{\infty} e^{-x} \left(1 + \cos x\right) dx$$

There are lots of ways to do this. Grading will be based on how well you explain your reasoning.

One way to do this is to compute the integral. We will do it using the comparison test. We have

$$1 + \cos x \leq 2$$

So

$$e^{-x} \left(1 + \cos x\right) \leq 2e^{-x}$$

We have

$$\int_0^{\infty} 2e^{-x} dx = lim_{b \to \infty} \int_0^b 2e^{-x} dx = lim_{b \to \infty} 2(1 - e^{-b})$$

which converes. So by the comparison test the oringal integral converges.