Math 129 (Fall '05) - Sample Exam 3 Solutions - Kennedy

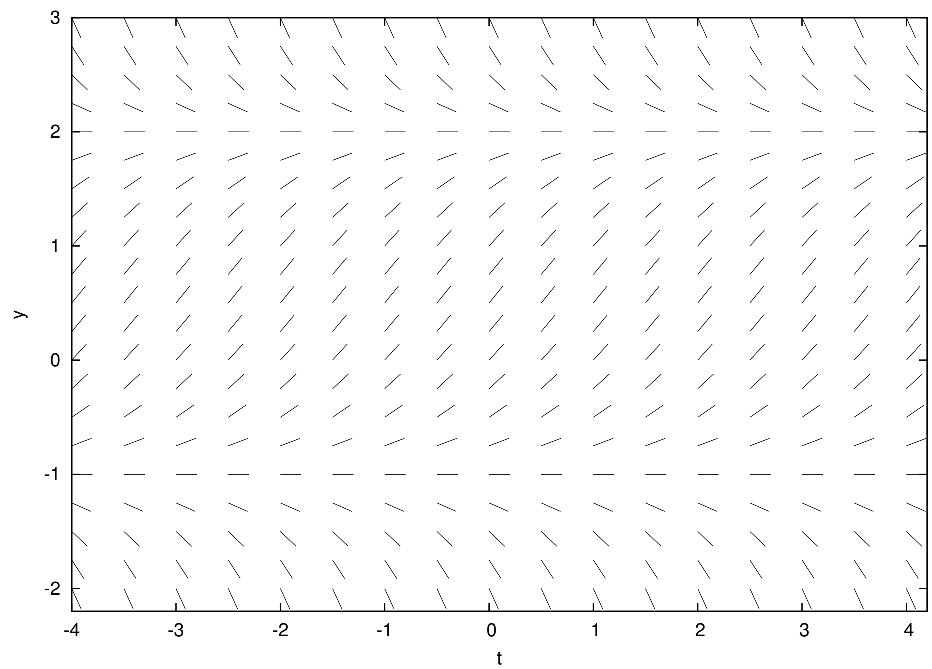
WARNING: This is a past exam I gave. You should not assume that if a certain topic does not appear here it will not appear on our exam.

1 The slope field for $\frac{dy}{dt} = 0.5(1+y)(2-y)$ is shown below.

(a) Sketch the solution which goes through the origin. (Your sketch should include the solution for negative values of t.)

(b) Consider the initial condition $y(0) = c$. For what values of c will the solution $y(t)$ converge to 2 as t goes to ∞ ?

Answer: It will converges to 2 for all $c > -1$.



2. Find the radius of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{2^n x^n}{n} \tag{1}$$

Answer:

$$\frac{2^{n+1}|x|^{n+1}}{n+1} \frac{n}{2^n|x|} = \frac{n}{n+1} 2|x| \tag{2}$$

Since $n/(n+1)$ converges to 1, the above converges to $2|x|$. So the series converges if $2|x| < 1$, i.e., for $-1/2 < x < 1/2$. So the radius of converges is $1/2$.

3. Find the Taylor series of e^x about $x = 1$. (NOTE that it is about $x = 1$, not about $x = 0$.) Your answer should go up to at least the $(x-1)^3$ term.

Answer: $f(x) = e^x$. All derivative $f^{(n)}(x)$ equal e^x. So $f^{(n)}(1) = e$ for all n. So the series is

$$e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \cdots + \frac{e}{n!}(x-1)^n + \cdots \tag{3}$$

4. For each of the following series state whether it converges or not and give the name of the test that gives your answer (integral test, comparison test, ratio test, alternating series test). In the case of the comparison test you should say what the series is that you are comparing with.

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \tag{4}$$

Apply the ratio test.

$$\frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} \tag{5}$$

This converges to $1/2$ as $n \to \infty$, so the ratio test says it converges since $1/2 < 1$.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tag{6}$$

Apply the integral test.

$$\int_1^{\infty} \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} [2\sqrt{b} - 2] \tag{7}$$

This diverges, so the series diverges.

$$\sum_{n=1}^{\infty} \frac{1+\cos(n)}{e^n} \tag{8}$$

Apply the comparison test. $1+\cos(n)$ is always between 0 and 2, so

$$0 \leq \frac{1+\cos(n)}{e^n} \leq \frac{2}{e^n} \tag{9}$$

Now $\sum \frac{2}{e^n}$ is geometric series with $r = 1/e < 1$ so it converges. So by the comparison test the original series does too.

5. Consider the differential equation and initial condition.

$$\frac{dy}{dt} = te^y, \quad y(0) = 0 \tag{10}$$

(a) Use Euler's method with $\Delta t = 0.1$ to find $y(1)$.

Answer: My calculator Euler program gives 0.56539.

(b) Find the exact solution of the differential equation. As a check on your answer, compare your exact value for $y(1)$ with your answer in (a).

Answer: Use separation of variables:

$$e^{-y}\,dy = t\,dt \tag{11}$$

Integrate this:

$$-e^{-y} = \frac{1}{2}t^2 + C \tag{12}$$

Using the initial condition $y(0) = 0$ we find that $-1 = C$. So

$$-e^{-y} = \frac{1}{2}t^2 - 1 \tag{13}$$

$$e^{-y} = \frac{-1}{2}t^2 + 1 \tag{14}$$

$$-y = \ln[\frac{-1}{2}t^2 + 1] \tag{15}$$

$$y = -\ln[1 - \frac{1}{2}t^2] \tag{16}$$

This gives $y(1) = -ln(1-1/2) = 0.693147$. This is not very close to the answer in (a), but that is because the step size is fairly large and Euler is a crude method. With a step size of 100, Euler gives 0.67848.

6. (a) Approximate $\sin(0.2)$ using a fifth-degree Taylor polynomial about $x = 0$ for $\sin(x)$.

Answer: The fifth order Taylor polynomial is $x - x^3/3! + x^5/5!$. Substituting $x = 0.2$ in this gives 0.198669.

(b) Estimate the magnitude of the error in your approximation.

Answer: The sixth derivative of $\sin(x)$ is $-\sin(x)$. So

$$M = \max_{[0,0.2]} |-\sin(x)| = \max_{[0,0.2]} \sin(x) = \sin(0.2) \tag{17}$$

This is certainly less than 1 which says the error is at most

$$\frac{1(0.2)^6}{6!} = 8.88 \times 10^{-8} \tag{18}$$

A better bound is to say $M = \sin(0.2)$ which then gives an error bound of

$$\frac{\sin(0.2)(0.2)^6}{6!} = 1.77 \times 10^{-8} \tag{19}$$

(c) Now approximate $\sin(0.2)$ using a sixth-degree Taylor polynomial about $x = 0$ for $\sin(x)$ and estimate the magnitude of the error in your approximation.

Answer: The sixth order Taylor polynomial is the same as the fifth order, so the estimate of $\sin(0.2)$ is the same. For the error bound, we now take the seventh derivative of $\sin(x)$ which is $-\cos(x)$. Then M is the max of $|\cos(x)|$ between 0 and 0.2. This is 1. So we get the bound

$$\frac{1(0.2)^7}{7!} = 2.54 \times 10^{-9} \tag{20}$$

7. (a) Find the Taylor series of $\tan^{-1}(x)$ about $x = 0$. Hint : what is the integral of $1/(1 + x^2)$?

Answer: Geometric series say

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots \tag{21}$$

So

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \cdots \tag{22}$$

Integrate this

$$\tan^{-1}(x) = x - x^3/3 + x^5/5 - x^7/7 \cdots \tag{23}$$

(b) Use your answer to (a) to find the value of the tenth derivative of $\tan^{-1}(x)$ at $x = 0$.

Answer Let D denote the tenth derivative evaluated at $x = 0$. Then the coeffecient of x^{10} in the Taylor series is $D/10!$. Since the coef of x^{10} is 0, the tenth derivative is zero.

Suppose I had asked for the ninth derivative at 0. Let D denote it. The coef of x^9 is $1/9$, so $D/9! = 1/9$. So $D = 9!/9 = 8! = 40320$.