Great Issues of Our Time: The Quadratic Formula

William McCallum

Tucson Teachers’ Circle, March 26, 2008
A proposal to eliminate quadratic equations

Terry Bladen, president of the National Association of Schoolmasters Union of Women Teachers:

"Pupils should be numerate but numeracy can be divorced from mathematics. How often do the majority of people need or use mathematical concepts once they have left school?"

[William McCallum, Great Issues of Our Time: The Quadratic Formula]
Terry Bladen, president of the National Association of Schoolmasters Union of Women Teachers:

“pupils should be numerate . . . but numeracy can be divorced from mathematics. . . . How often do the majority of people need or use mathematical concepts once they have left school?”
A proposal to eliminate quadratic equations

Terry Bladen, president of the National Association of Schoolmasters Union of Women Teachers:

“pupils should be numerate . . . but numeracy can be divorced from mathematics. . . . How often do the majority of people need or use mathematical concepts once they have left school?”

[He advocated] allowing them to drop advanced concepts such as quadratic equations and trigonometry at the age of 14.
The proposal is debated in parliament.

Tony McWalter, Labour MP

---

"A quadratic equation is not like a bleak room devoid of furniture in which one is asked to squat — it is a door to a room full of the unparalleled riches of human intellectual achievement — if you do not go through that door, much that passes for human wisdom will be forever denied you."

-- Eleanor Laing, Conservative MP

"Oh dear — would like to have support from elsewhere as well."

William McCallum
Tony McWalter, Labour MP

“A quadratic equation is not like a bleak room, devoid of furniture, in which one is asked to squat. It is a door to a room full of the unparalleled riches of human intellectual achievement. If you do not go through that door . . . much that passes for human wisdom will be forever denied you.”
Tony McWalter, Labour MP

“A quadratic equation is not like a bleak room, devoid of furniture, in which one is asked to squat. It is a door to a room full of the unparalleled riches of human intellectual achievement. If you do not go through that door . . . much that passes for human wisdom will be forever denied you.”

“Hear, hear”—Eleanor Laing, Conservative MP
The proposal is debated in parliament

26 June 2003

Tony McWalter, Labour MP

“A quadratic equation is not like a bleak room, devoid of furniture, in which one is asked to squat. It is a door to a room full of the unparalleled riches of human intellectual achievement. If you do not go through that door . . . much that passes for human wisdom will be forever denied you.”

“Hear, hear”—Eleanor Laing, Conservative MP
Tony McWalter, Labour MP

“A quadratic equation is not like a bleak room, devoid of furniture, in which one is asked to squat. It is a door to a room full of the unparalleled riches of human intellectual achievement. If you do not go through that door . . . much that passes for human wisdom will be forever denied you.”

“Hear, hear”—Eleanor Laing, Conservative MP

“Oh dear. I would like to have support from elsewhere as well.”
The minister replies

26 June 2003

Alan Johnson, Minister

The United Kingdom Parliament

William McCallum

Great Issues of Our Time: The Quadratic Formula
Alan Johnson, Minister

“In preparing for this debate, the DFES conducted a straw poll involving a 16-year-old who had just sat maths GCSE, a head of maths and an experienced chemical engineer.”
Alan Johnson, Minister

“In preparing for this debate, the DFES conducted a straw poll involving a 16-year-old who had just sat maths GCSE, a head of maths and an experienced chemical engineer.”

The 16-year-old thought that quadratic equations were logical and fairly straightforward because ‘you substitute stuff into a formula’. . . .
The minister replies

Alan Johnson, Minister

“In preparing for this debate, the DFES conducted a straw poll involving a 16-year-old who had just sat maths GCSE, a head of maths and an experienced chemical engineer.”

- The 16-year-old thought that quadratic equations were logical and fairly straightforward because ‘you substitute stuff into a formula’. . . .

- The head of maths said that quadratic equations formed an important step in students’ ability to solve equations, . . .
In preparing for this debate, the DFES conducted a straw poll involving a 16-year-old who had just sat maths GCSE, a head of maths and an experienced chemical engineer.

The 16-year-old thought that quadratic equations were logical and fairly straightforward because ‘you substitute stuff into a formula’. . . .

The head of maths said that quadratic equations formed an important step in students’ ability to solve equations, . . .

The engineer said that he did not use quadratic equations now, but had in the past . . .
... what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. ... Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.
The manner of solving this type of equation is to take one-half of the roots just mentioned. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

Exercise

What equation is al-Khwarizmi talking about here?

» Answer  » Skip
what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

\[ x^2 + 10x = 39 \]
... what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. ... Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

\[ x^2 + 10x = 39 \]
... what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. ... Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

\[ x^2 + 10x = 39 \]

\[ x^2 + 10x + 25 = 39 + 25 = 64 \]
... what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

\[
x^2 + 10x = 39
\]

\[
x^2 + 10x + 25 = 39 + 25 = 64
\]
...what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

\[ x^2 + 10x = 39 \]
\[ x^2 + 10x + 25 = 39 + 25 = 64 \]
\[ x + 5 = 8 \]
\[ x = 3 \]
Al-Khwarizmi’s geometric proof of his method

\[
x^2 + 10x = 39
\]

\[
x^2 + 10x + 25 = 39 + 25 = 64
\]

\[
x + 5 = 8
\]

\[
x = 3
\]
Al-Khwarizmi’s geometric proof of his method

\[ x^2 + 10x = 39 \]
\[ x^2 + 10x + 25 = 39 + 25 = 64 \]
\[ x + 5 = 8 \]
\[ x = 3 \]
Al-Khwarizmi’s geometric proof of his method

\[ x^2 + 10x = 39 \]
\[ x^2 + 10x + 25 = 39 + 25 = 64 \]
\[ x + 5 = 8 \]
\[ x = 3 \]
Al-Khwarizmi’s geometric proof of his method

\[ x^2 + 10x = 39 \]
\[ x^2 + 10x + 25 = 39 + 25 = 64 \]
\[ x + 5 = 8 \]
\[ x = 3 \]
Al-Khwarizmi’s geometric proof of his method

\[ x^2 + 10x = 39 \]
\[ x^2 + 10x + 25 = 39 + 25 = 64 \]
\[ x + 5 = 8 \]
\[ x = 3 \]
Al-Khwarizmi’s geometric proof of his method

\[
x^2 + 10x = 39
\]
\[
x^2 + 10x + 25 = 39 + 25 = 64
\]
\[
x + 5 = 8
\]
\[
x = 3
\]

\[
\begin{array}{|c|c|}
\hline
5x & x^2 \\
\hline
25 & 5x \\
\hline
\end{array}
\]

Exercise

Draw a diagram that illustrates the solution of the equation

\[
x^2 = 39 + 10x.
\]
Al-Khwarizmi’s geometric proof of his method

\[ x^2 + 10x = 39 \]
\[ x^2 + 10x + 25 = 39 + 25 = 64 \]
\[ x + 5 = 8 \]
\[ x = 3 \]

\[ x^2 = 39 + 10x \]
Al-Khwarizmi’s geometric proof of his method

\[ x^2 + 10x = 39 \]
\[ x^2 + 10x + 25 = 39 + 25 = 64 \]
\[ x + 5 = 8 \]
\[ x = 3 \]

\[ x^2 = 39 + 10x \]
\[ x^2 + 25 = 64 + 10x \]

There is a better way.

William McCallum
Great Issues of Our Time: The Quadratic Formula
Al-Khwarizmi’s geometric proof of his method

\[ x^2 + 10x = 39 \]
\[ x^2 + 10x + 25 = 39 + 25 = 64 \]
\[ x + 5 = 8 \]
\[ x = 3 \]

\[ x^2 = 39 + 10x \]
\[ x^2 + 25 = 64 + 10x \]

There is a better way.
Al-Khwarizmi’s geometric proof of his method

\[
x^2 + 10x = 39
\]
\[
x^2 + 10x + 25 = 39 + 25 = 64
\]
\[
x + 5 = 8
\]
\[
x = 3
\]

\[
x^2 = 39 + 10x
\]
\[
x^2 + 25 = 64 + 10x
\]

There is a better way.
Al-Khwarizmi’s geometric proof of his method

\[ x^2 + 10x = 39 \]
\[ x^2 + 10x + 25 = 39 + 25 = 64 \]
\[ x + 5 = 8 \]
\[ x = 3 \]

\[ x^2 = 39 + 10x \]
\[ x^2 + 25 = 64 + 10x \]
\[ x - 5 = 8 \]
\[ x = 13 \]

There is a better way.
Problems were phrased in terms of areas, weights, money.
Only positive solutions are allowed

Problems were phrased in terms of areas, weights, money.

Al-Khwarizmi’s method guarantees a unique solution to the equations

\[ x^2 + Cx = N \]
\[ x^2 = Cx + N \]

where \( N \) and \( C \) are positive numbers.
Only positive solutions are allowed

Problems were phrased in terms of areas, weights, money.
Al-Khwarizmi’s method guarantees a unique solution to the equations

\[
x^2 + Cx = N
\]
\[
x^2 = Cx + N
\]

where \(N\) and \(C\) are positive numbers.
So each equation must have a negative solution as well.
Problems were phrased in terms of areas, weights, money. Al-Khwarizmi’s method guarantees a unique solution to the equations

\[ x^2 + Cx = N \]
\[ x^2 = Cx + N \]

where \( N \) and \( C \) are positive numbers.

So each equation must have a negative solution as well.

Imaginary solutions were far in the future.
Completing the square the modern way

\[ x^2 + bx + c = 0 \]
Completing the square the modern way

\[ x^2 + bx + c = 0 \]

\[ x^2 + bx = -c \]
Completing the square the modern way

\[ x^2 + bx + c = 0 \]
\[ x^2 + bx = -c \]
\[ x^2 + bx + \left(\frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2 \]
Completing the square the modern way

\[ x^2 + bx + c = 0 \]

\[ x^2 + bx = -c \]

\[ x^2 + bx + \left(\frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2 \]

\[ \left(x + \frac{b}{2}\right)^2 = \frac{b^2 - 4c}{4} \]
Completing the square the modern way

\[ x^2 + bx + c = 0 \]

\[ x^2 + bx = -c \]

\[ x^2 + bx + \left( \frac{b}{2} \right)^2 = -c + \left( \frac{b}{2} \right)^2 \]

\[ \left( x + \frac{b}{2} \right)^2 = \frac{b^2 - 4c}{4} \]

\[ x + \frac{b}{2} = \pm \frac{\sqrt{b^2 - 4c}}{2} \]
Completing the square the modern way

\[ x^2 + bx + c = 0 \]

\[ x^2 + bx = -c \]

\[ x^2 + bx + \left(\frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2 \]

\[ \left(x + \frac{b}{2}\right)^2 = \frac{b^2 - 4c}{4} \]

\[ x + \frac{b}{2} = \pm \frac{\sqrt{b^2 - 4c}}{2} \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \]
Completing the square the modern way

\[ ax^2 + bx + c = 0 \]

\[ x^2 + \frac{b}{a}x = \frac{-c}{a} \]

\[ x^2 + \frac{b}{a}x + \left( \frac{b}{2a} \right)^2 = \frac{-c}{a} + \left( \frac{b}{2a} \right)^2 \]

\[ \left( x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2} \]

\[ x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
The quadratic formula

A number \( x \) satisfies

\[ ax^2 + bx + c = 0, \quad a \neq 0, \]

if, and only if,

\[ x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \]
A number $x$ satisfies

$$ax^2 + bx + c = 0, \quad a \neq 0,$$

if, and only if,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

### Exercise

Give some simple conditions on the coefficients for a quadratic equation to have

(a) two real roots
(b) two positive roots.
The quadratic formula

A number \( x \) satisfies

\[
ax^2 + bx + c = 0, \quad a \neq 0,
\]

if, and only if,

\[
x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.
\]

Answer

(a) Make \( a \) and \( c \) have opposite signs.
A number $x$ satisfies

$$ax^2 + bx + c = 0, \quad a \neq 0,$$

if, and only if,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$
Another way of solving quadratic equations

For all numbers $x$, 

$$x^2 + 10x - 39 = (x - 3)(x + 13) = 0.$$
Another way of solving quadratic equations

For all numbers $x$,

$$x^2 + 10x - 39 = (x - 3)(x + 13) = 0.$$  

So

$$x^2 + 10x - 39 = 0$$

if, and only if,

$$(x - 3)(x + 13) = 0.$$
Another way of solving quadratic equations

For all numbers $x$,

$$x^2 + 10x - 39 = (x - 3)(x + 13) = 0.$$

So

$$x^2 + 10x - 39 = 0$$

if, and only if,

$$(x - 3)(x + 13) = 0.$$

The product of two numbers is zero if, and only if, one of them is zero, so either

$$x - 3 = 0 \quad \text{or} \quad x + 13 = 0.$$

That is, $x = 3$ or $x = -13$. 
Theorem (Viete’s formulae)

The numbers $r$ and $s$ are the solutions to

$$x^2 + bx + c = 0$$

if, and only if,

$$r + s = -b \quad \text{and} \quad rs = c.$$
Viete’s formulae

Theorem (Viete’s formulae)

The numbers $r$ and $s$ are the solutions to

$$x^2 + bx + c = 0$$

if, and only if,

$$r + s = -b \quad \text{and} \quad rs = c.$$

Proof of the “if” part.

If $r + s = -b$ and $rs = c$, then

$$x^2 + bx + c = (x - r)(x - s).$$

Then the argument goes as in the previous slide.
Theorem (Viete’s formulae)

The numbers $r$ and $s$ are the solutions to

$$x^2 + bx + c = 0$$

if, and only if,

$$r + s = -b \quad \text{and} \quad rs = c.$$ 

Exercise

How do you prove the “only if” part? That is, if $r$ and $s$ are the solutions to $x^2 + bx + c$, then $r + s = -b$ and $rs = c$. 

▶ Answer  ▶ Skip
Theorem (Viete’s formulae)

The numbers \( r \) and \( s \) are the solutions to

\[
x^2 + bx + c = 0
\]

if, and only if,

\[
 r + s = -b \quad \text{and} \quad rs = c.
\]

Proof of “only if” part

Divide \( x - r \) into \( x^2 + bx + c \), so

\[
x^2 + bx + c = (x - r)q(x) + R.
\]

Putting \( x = r \) we get \( R = 0 \). Then \( q(x) = x - t \) for some \( t \), and the only possibility is \( t = s \).
If

\[ x^2 + bx + c = 0 \]

then let

\[ r = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4c}}{2} \]
Viete’s formulae and the quadratic formula

If

\[ x^2 + bx + c = 0 \]

then let

\[ r = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4c}}{2} \]

**Exercise**

Give an explanation, purely in terms of the structure of the expressions, of why these two numbers satisfy

\[ r + s = -b \quad \text{and} \quad rs = c. \]
If

\[ x^2 + bx + c = 0 \]

then let

\[ r = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4c}}{2} \]

**Answer**

When you add \( r \) and \( s \), the plus and minus signs cancel.
Viete’s formulae and the quadratic formula

If

\[ x^2 + bx + c = 0 \]

then let

\[ r = \frac{-b}{2} \quad \text{and} \quad s = \frac{-b}{2} \]

Answer

When you add \( r \) and \( s \), the plus and minus signs cancel.
Viete’s formulae and the quadratic formula

If

\[ x^2 + bx + c = 0 \]

then let

\[
\begin{align*}
  r &= \boxed{} + \boxed{\frac{g}{2}} \\
  s &= \boxed{} - \boxed{\frac{g}{2}}
\end{align*}
\]

\[ r = \left( -\frac{b}{2} \right) + \frac{\sqrt{b^2 - 4c}}{2} \quad \text{and} \quad s = \left( -\frac{b}{2} \right) - \frac{\sqrt{b^2 - 4c}}{2} \]

Answer

When you add \( r \) and \( s \), the plus and minus signs cancel.
When you multiply \( r \) and \( s \), you get the difference of two squares in the numerator,

\[
(-b)^2 - (\sqrt{b^2 - ac})^2 = b^2 - (b^2 - 4c) = 4c.
\]
The quadratic formula in the 17th century

From the Oxford Museum of History of Science (Stephen Johnston, photo Bluebridge Farm Studio)
The quadratic formula in the 17th century

Exercise

What is going on here?
What is going on here?

\[ z + Cr = N : \sqrt{qu} : \frac{Cq}{4} + N : - \frac{C}{2} = r \]

\[ Cr - z = N : \left\{ \frac{C}{2} + \sqrt{qu} : \right\} \frac{Cq}{4} - N : = r \]

\[ z - Cr = N : \sqrt{qu} : \frac{Cq}{4} + N : + \frac{C}{2} = r \]
What is going on here?

\[ x^2 + Cx = N : \sqrt{qu} : \frac{Cq}{4} + N : - \frac{C}{2} = x \]

\[ Cx - x^2 = N : \left\{ \frac{C}{2} + \sqrt{qu} : \right\} \frac{Cq}{4} - N : = x \]

\[ x^2 - Cx = N : \sqrt{qu} : \frac{Cq}{4} + N : + \frac{C}{2} = x \]
What is going on here?

\[ x^2 + Cx = N : \sqrt{qu : \frac{C^2}{4}} + N : - \frac{C}{2} = x \]

\[ Cx - x^2 = N : \left\{ \frac{C}{2} + \sqrt{qu : \frac{C^2}{4}} \right\} \frac{C^2}{4} - N : = x \]

\[ x^2 - Cx = N : \sqrt{qu : \frac{C^2}{4}} + N : + \frac{C}{2} = x \]
What is going on here?

\[ x^2 + Cx = N, \quad \sqrt{qu} : \frac{C^2}{4} + N : - \frac{C}{2} = x \]

\[ Cx - x^2 = N, \quad \left\{ \frac{C}{2} + \sqrt{qu} : \frac{C^2}{4} - N : = x \right\} \]

\[ x^2 - Cx = N, \quad \sqrt{qu} : \frac{C^2}{4} + N : + \frac{C}{2} = x \]
What is going on here?

\[ x^2 + Cx = N, \quad \sqrt{qu} : \frac{C^2}{4} + N : - \frac{C}{2} = x \]

\[ Cx - x^2 = N, \quad \frac{C}{2} \pm \sqrt{qu} : \frac{C^2}{4} - N : = x \]

\[ x^2 - Cx = N, \quad \sqrt{qu} : \frac{C^2}{4} + N : + \frac{C}{2} = x \]
What is going on here?

\[ x^2 + Cx = N, \quad \sqrt{\frac{C^2}{4} + N - \frac{C}{2}} = x \]

\[ Cx - x^2 = N, \quad \frac{C}{2} \pm \sqrt{\frac{C^2}{4} - N} = x \]

\[ x^2 - Cx = N, \quad \sqrt{\frac{C^2}{4} + N + \frac{C}{2}} = x \]
Example

If

\[ x^2 + 10x - 39 = 0, \]

then

\[ x = \frac{-10 \pm \sqrt{256}}{2} = -5 \pm \sqrt{64} = 3, -13. \]

Example

If

\[ x^2 - 10x + 9 = 0 \]

then

\[ x = \frac{10 \pm \sqrt{10^2 - 4 \times 9}}{2} = \frac{10 \pm \sqrt{64}}{2} = 5 \pm 4 = 1, 9. \]
Example

If

\[ x^2 + 10x - 39 = 0, \]

then

\[ x = \frac{-10 \pm \sqrt{256}}{2} = -5 \pm \sqrt{64} = 3, -13. \]

Example

If

\[ x^2 - 10x + 9 = 0 \]

then

\[ x = \frac{10 \pm \sqrt{10^2 - 4 \times 9}}{2} = \frac{10 \pm \sqrt{64}}{2} = 5 \pm 4 = 1, 9. \]
Example

If

\[ x^2 + 10x - 39 = 0, \]

then

\[ x = \frac{-10 \pm \sqrt{256}}{2} = -5 \pm \sqrt{64} = 3, -13. \]

Example

If

\[ x^2 - 10x + 9 = 0 \]

then

\[ x = \frac{10 \pm \sqrt{10^2 - 4 \times 9}}{2} = \frac{10 \pm \sqrt{64}}{2} = 5 \pm 4 = 1, 9. \]