THE GAN–GROSS–PRASAD CONJECTURE FOR $\mathrm{U}(n) \times \mathrm{U}(n)$

HANG XUE

ABSTRACT. We prove the Gan–Gross–Prasad conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ under some local conditions using a relative trace formula. We deduce some new cases of the Gan–Gross–Prasad conjecture for $\mathrm{U}(n+1) \times \mathrm{U}(n)$ from the case of $\mathrm{U}(n) \times \mathrm{U}(n)$.

CONTENTS

1. INTRODUCTION

1.1. **Gan–Gross–Prasad conjecture.** In 1990's, Gross and Prasad proposed in a series of papers some fascinating conjectures on the restriction of automorphic representations of the special orthogonal groups to smaller special orthogonal groups. Recently, together with Gan, they generalized these conjectures to all the classical groups [GGP2012]. In the case of unitary groups, these conjectures relate the period integral of automorphic forms on $\mathrm{U}(n) \times \mathrm{U}(m)$ to the central value of some Rankin–Selberg L-function.

Recently, Jacquet–Rallis [JR2011] proposed a new approach towards the Gan–Gross–Prasad conjecture (GGP conjecture for short) for $\mathrm{U}(n+1) \times \mathrm{U}(n)$. It is based on some relative trace formulae. Soon afterwards, the relevant fundamental lemma was proved by Yun [Yun2011]. Zhang [Zha] then proved a simple version of the Jacquet–Rallis trace formula by establishing the existence of the smooth transfer. Zhang then deduced the GGP conjecture for $\mathrm{U}(n+1) \times \mathrm{U}(n)$ under some local conditions from the Jacquet–Rallis trace formula.

In the other direction, Liu [Liu] generalized the relative trace formula approach of Jacquet–Rallis to all pairs of unitary groups, in particular for the pair $\mathrm{U}(n) \times \mathrm{U}(n)$. Several important results are also obtained by him. For example, the fundamental lemma for $\mathrm{U}(n) \times \mathrm{U}(n)$ has been completely proved.

This paper grows out of an attempt to understand the important work by Liu [Liu] and Zhang [Zha]. The result of Liu suggests a strong connection between the relative trace formulae for $\mathrm{U}(n) \times \mathrm{U}(n)$ and for $\mathrm{U}(n+1) \times \mathrm{U}(n)$. In fact, the fundamental lemma for the former is reduced to the later. We shall see in this paper that the smooth transfer for $\mathrm{U}(n) \times \mathrm{U}(n)$ can also be reduced to the one proposed by Jacquet–Rallis, which has been proved by Zhang in [Zha].

To state our results, we need to introduce some notation. Let k' be a number field and k a quadratic field extension of k'. We denote by τ the non-trivial element in the Galois group $\mathrm{Gal}(k/k')$, $\mathbb{A}'$ and $\mathbb{A}$ rings of adeles respectively, η the quadratic character of $k'^\times \backslash \mathbb{A}'^\times$ associated to k/k' by global class field theory, μ a character of $k^\times \backslash \mathbb{A}^\times$ with $\mu|_{\mathbb{A}'^\times} = \eta$. Fix a nontrivial additive character $\psi : k' \backslash \mathbb{A}' \to \mathbb{C}^\times$.

Let $(V, \langle -, - \rangle)$ be a Hermitian space over k, $\mathrm{U}(V)$ the unitary group attached to V over k'. Fix a purely imaginary element $j \in k$. We define $\mathrm{Tr}(x) = x + x^{\tau}$ and $\widetilde{\mathrm{Tr}}(x) = \mathrm{Tr}(jx)$. The Weil restriction $\mathrm{Res}_{k/k'} V^{\vee}$ is a symplectic space under the pairing $\widetilde{\mathrm{Tr}}(\langle -, - \rangle)$. There is a global Weil representation $\omega_{\psi, \mu}$ of the unitary group $\mathrm{U}(V)(\mathbb{A}')$. The Weil representation is realized on the Schwartz space $\mathcal{S}(\mathbf{L}(\mathbb{A}'))$ where $\mathbf{L} + \mathbf{L}^{\vee} = \mathrm{Res}_{k/k'} V^{\vee}$ is a polarization with $\mathbf{L}, \mathbf{L}^{\vee} \subset \mathrm{Res}_{k/k'} V^{\vee}$ being Lagrangian subspaces. For any $\phi \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$, define the theta series

$$\theta_{\psi, \mu}(g, \phi) = \sum_{v \in \mathbf{L}(k')} \omega_{\psi, \mu}(g) \phi(v).$$

This is an automorphic form on $\mathrm{U}(V)(\mathbb{A}')$.

Let π be an irreducible cuspidal automorphic representation of $\mathrm{U}(V)(\mathbb{A}') \times \mathrm{U}(V)(\mathbb{A}')$. An element in

$$\mathrm{Hom}_{\mathrm{U}(V)(\mathbb{A}')} \left(\pi \otimes \overline{\omega_{\psi, \mu}}, \mathbb{C} \right)$$

is called a (global) Fourier–Jacobi period. This space has dimension at most one, c.f. [Sun2012].

For $\varphi \in \pi$ and $\phi \in \mathcal{S}(\mathbf{L}(\mathbb{A}))$, we define the Fourier–Jacobi integral

$$(1.1.1) \qquad \mathcal{FJ}_{\psi, \mu}(\varphi, \phi) = \int_{\mathrm{U}(V)(k') \backslash \mathrm{U}(V)(\mathbb{A}')} \varphi(\iota(g)) \theta_{\psi^{-1}, \mu^{-1}}(g, \phi) \mathrm{d}g.$$

Here $\iota : \mathrm{U}(V) \to \mathrm{U}(V) \times \mathrm{U}(V)$ is the diagonal embedding. It is clear that $\mathcal{FJ}_{\psi, \mu}(\varphi, \phi)$ is a Fourier–Jacobi period of π. The question is: when is it non-zero? The GGP conjecture predicts that the nonvanishing of the Fourier–Jacobi period is related to the central value of the L-function.

Let $\pi = \otimes'_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{U}(V)(\mathbb{A}')$. Let $G = \mathrm{Res}_{k/k'} \mathrm{GL}(V)$. We assume the existence of the weak base change $\mathrm{BC}(\pi)$ of π to $G(\mathbb{A}')$ in this paper, c.f. [HL2004]. If there is a split place v of k' such that π_v is supercuspidal, then $\mathrm{BC}(\pi)$ is again cuspidal.

Suppose V and V' are two Hermitian spaces of the same dimension. Then the unitary groups $\mathrm{U}(V)$ and $\mathrm{U}(V')$ are identified at all but finitely many places. Let σ and σ' be automorphic representations of $\mathrm{U}(V)(\mathbb{A}')$ and $\mathrm{U}(V')(\mathbb{A}')$ respectively. Then σ and σ' are said to be nearly equivalent if the local components σ_v and σ'_v are isomorphic for all but finitely many places. Note that if σ and σ' are nearly equivalent, then their weak base change are isomorphic by strong multiplicity one theorem of the general linear group. Moreover, the local components σ_v and σ'_v are isomorphic for all the split places v of k'.

Let $\pi = \pi_1 \boxtimes \pi_2$ be an irreducible cuspidal automorphic representation of $\mathrm{U}(V)(\mathbb{A}') \times \mathrm{U}(V)(\mathbb{A}')$ where π_1 and π_2 are irreducible cuspidal automorphic representations of $\mathrm{U}(V)(\mathbb{A}')$.

Theorem 1.1.1. *Assume*

(1) *There are two non-archimedean places v_1 and v_2 of k' that split in k, such that the local components $\pi_{1, v_i}, \pi_{2, v_i}$ of π are all supercuspidal ($i = 1, 2$).*

(2) *All the archimedean places of k' split in k.*

Then the following are equivalent.

(1) $L(\frac{1}{2}, \mathrm{BC}(\pi_1) \times \mathrm{BC}(\pi_2) \otimes \mu^{-1}) \neq 0$.

(2) *There is an n-dimensional Hermitian space V', and a representation π' of $\mathrm{U}(V')(\mathbb{A}') \times \mathrm{U}(V')(\mathbb{A}')$, which is nearly equivalent to π, such that the Fourier–Jacobi period of π' is not identically zero.*

Remark 1.1.2. If $k = k' \times k'$, by the work of [JPSS1983], this theorem is known in complete generality, i.e. with no local conditions. Furthermore, the precise formula relating the Fourier–Jacobi model and the central value of the L-function is also known in this case.

Remark 1.1.3. From the local GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ [GGP2012, § 17], the Hermitian space V' in (2) of Theorem 1.1.1 should be unique (up to isomorphism). Gan and Ichino [GI] has proved the local GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ in the non-archimedean case by reducing it to the local GGP conjecture for $\mathrm{U}(n+1) \times \mathrm{U}(n)$ which has been proved by Beuzart-Plessis [BP].

The proof of this theorem is based on a simple version of the relative trace formula of Liu [Liu]. The argument follows the same line as in Zhang [Zha]. In particular, several restrictions on the test functions are imposed so that lots of terms on both sides of the relative trace formula vanish.

As already noted by [GGP2012, § 14], the GGP conjectures for $U(n) \times U(n)$ and $U(n+1) \times U(n)$ are closely related. In the last section of this paper, we apply Theorem 1.1.1 to deduce some new cases of GGP conjecture for $U(n+1) \times U(n)$. The main tool that we need is the theta correspondence for unitary groups. We need to assume more properties of the weak base change. We assume that the weak base change is isobaric. We also assume that it respects the L-functions defined by the doubling method. See Hypothesis BC in Section 8.1. Let V and W be two Hermitian spaces of dimension $n+1$ and n respectively. Assume $W \subset V$. The group $U(W)$ embeds in $U(V) \times U(W)$ via

$$\iota : U(W) \to U(V) \times U(W), \quad g \mapsto \left(\begin{pmatrix} g & \\ & 1 \end{pmatrix}, g \right).$$

Let $\pi = \pi_V \boxtimes \pi_W$ be an irreducible cuspidal automorphic representation of $U(V)(\mathbb{A}') \times U(W)(\mathbb{A}')$. The Bessel period of π is defined by

$$\mathcal{B}(\varphi) = \int_{U(W)(k') \backslash U(W)(\mathbb{A}')} \varphi(\iota(g)) \mathrm{d}g.$$

Theorem 1.1.4. *Assume the following conditions.*

(1) *All the archimedean places v of k' split in k.*
(2) *There is a Hermitian space W_0, and irreducible cuspidal automorphic representaions π_0 and σ_0 of $U(W_0)(\mathbb{A}')$, such that π_V is the theta lift of π_0 and π_W is the theta lift of σ_0. Moreover, at two split non-archimdean places v_1 and v_2 of F, the representations π_{0,v_i} and σ_{0,v_i}, $(i = 1, 2)$ are all supercuspidal.*

Then the following statements are equivalent.

(1) *The L-function $L(s, BC(\pi_V) \times BC(\pi_W))$ does not vanish at $s = \frac{1}{2}$.*
(2) *There are Hermitian spaces W' and V' of dimension n and $n+1$ respectively, and an irreducible cuspidal automorphic representation π' of $U(V')(\mathbb{A}') \times U(W')(\mathbb{A}')$ that is nearly equivalent to π, such that the Bessel period integral is not identically zero on π'.*

Remark 1.1.5. The GGP conjecture for $U(n+1) \times U(n)$ is proved in [Zha] under some local conditions. In particular, the local conditions of [Zha, Theorem 1.1] implies that the representation π is stable, in the sense that the weak base change of π is still cuspidal. In our theorem, the weak base change of π_V is not cuspidal. Therefore this case cannot be dealt with directly using the simple version of the relative trace formula of Jacquet–Rallis as in [Zha]. The relative trace formula of Jacquet–Rallis cannot be applied to this situation until more information on the spectral side is available.

1.2. **Notation and convention.** Throughout this paper, we use the following notation and make the following assumptions.

- Let k' be a local field or a number field and k a quadratic etale algebra over k'. Then either $k = k' \times k'$ or k is a quadratic field extension of k'. Let $\mathfrak{o}$ (resp. $\mathfrak{o}'$) be the ring of integers in k (resp. k'). If $k \neq k' \times k'$, we let τ be the nontrivial element in $\mathrm{Gal}(k/k')$. If $k = k' \times k'$, we let τ be the automorphism of k switching two factors.
- We say that we are in the local situation if k' is a local field. In this case, we let $\eta : k'^\times \to \mathbb{C}^\times$ be the character associated to k/k' by class field theory. We take a character μ of $k^\times$ which satisfies $\mu|_{k'^\times} = \eta$. We fix $j \in k^\times$ a purely imaginary element in $k^\times$, i.e. $j^\tau = -j$. Let Tr be the trace of k/k', and $\widetilde{\mathrm{Tr}}(x) := \mathrm{Tr}(jx)$. Let $\psi' : k' \to \mathbb{C}^\times$ be a nontrivial additive character and $\psi = \psi' \circ \widetilde{\mathrm{Tr}}$. In the case $k = k' \times k'$, we sometimes write $k = k'_\circ \times k'_\bullet$ where the subscripts indicate the first and second factor.
- In the local situation, we say that we are in the unramified local situation if the following conditions holds. The extension k/k' is either an unramified quadratic field extension or $k = k' \times k'$. The purely imaginary element j belongs to $\mathfrak{o}^\times$. The conductor of ψ' (resp. μ) is $\mathfrak{o}$ (resp. $\mathfrak{o}^\times$).
- We say that we are in the global situation if k' is a number field. In this case, we always assume $k \neq k' \times k'$. Let $\mathbb{A}'$ (resp. $\mathbb{A}$) be the ring of adeles of k' (resp. k). We denote $\mathbb{A}'_\mathrm{f}$ (resp. $\mathbb{A}'_\infty$) the finite (resp. infinite) adele of $\mathbb{A}'$. Let $\eta : k'^\times \backslash \mathbb{A}'^\times \to \mathbb{C}^\times$ be the character associated to k/k' by class field theory. We take an automorphic character μ of $\mathbb{A}^\times$ which satisfies $\mu|_{\mathbb{A}'^\times} = \eta$. We fix

a purely imaginary element $j \in k^\times$. Let $\psi' : \mathbb{A}'/k' \to \mathbb{C}^\times$ be a non-trivial additive character and $\psi = \psi' \circ \mathrm{Tr}_{\mathbb{A}/\mathbb{A}'}$.

- In the global situation, let v be a place of k'. We always assume that v splits in k if v is archimedean. We denote by ψ'_v the v component of the additive character ψ'. Similar notation also applies to other global objects. If v splits in k, we denote the two places above v by $v_\circ$ and $v_\bullet$. Then $k_v = k'_{v_\circ} \times k'_{v_\bullet}$.

- We denote by $\mathrm{Mat}_{m,n}$ the affine group scheme of $m \times n$ matrices. For any commutative ring R, we denote $\mathrm{Mat}_{m,1}(R)$ (resp. $\mathrm{Mat}_{1,m}(R)$) by R^m (resp. R_m). We denote by $\mathbf{1}_n$ the $n \times n$ identity matrix.

- In either local or global situation, we introduce the following symmetric space. Let S_n be the algebraic variety over k' defined by

$$\mathsf{S}_n(R) = \{g \in \mathrm{GL}_n(R \otimes_{k'} k) \mid {}^t g g = \mathbf{1}_n\},$$

for any k' algebra R. There is a canonical morphism

$$\sigma : \mathrm{Res}_{k/k'} \mathrm{GL}_n \to \mathsf{S}_n, \quad g \mapsto g g^{\tau,-1},$$

which is surjective on the level of k'-points by Hilbert' Theorem 90.

- In either local or global situation, let $(V, \langle -, - \rangle_V)$ be a Hermitian space over k. We take the convention that $\langle -, - \rangle_V$ is anti-linear in the first variable and linear in the second variable. Let $V^\vee$ be the dual of V. The Hermitian form of V (resp. $V^\vee$) is given by a Hermitian matrix β (resp. ${}^t\beta$)) when we choose a basis of V (resp. the dual basis of $V^\vee$). Let $\mathsf{Herm}_n(k)$ be the set of isomorphism classes of hermitian spaces over k. Let $\mathrm{U}(V)$ be the corresponding unitary group over k'. If $k = k' \times k'$, then $\mathsf{Herm}_n(k)$ is a singleton and

$$\mathrm{U}(V) = \{(g_\circ, g_\bullet) \mid {}^t g_\circ g_\bullet = \mathbf{1}_n\}.$$

It is identified with GL_n via the projection to the second factor.

- Let G be a reductive group acting on an affine variety X. The categorical quotient exists and is denoted by $X /\!\!/ G$. An element $x \in X$ is called semisimple if the orbit is closed, and is called regular if the stablizer is of minimal dimension. The regular semisimple locus in X is denoted by X_{rss}.

- In the local situation, let X be an algebraic variety over k'. Denote by $\mathcal{C}_c^\infty(X(k'))$ the space of compactly supported function on $X(k')$. If k' is nonarchimedean, by a Schwartz function on $X(k')$, we mean a locally constant compactly supported function on $X(k')$. If k' is archimedean, a Schwartz function f is a smooth function such that Df is bounded for any differential operator D on $X(k')$. The space of Schwartz functions is denoted by $\mathcal{S}(X(k'))$.

- In the global situation, let X be an algebraic variety over k'. In all the examples we encounter, there is a finite set S of places v of k', so that X is defined over $\mathrm{Spec}\, \mathfrak{o}_{k',v}$ for all $v \notin S$. Let $\mathbf{1}_{X(\mathfrak{o}_{k',v})}$ be the characteristic function of $X(\mathfrak{o}_{k',v})$. This is a Schwartz function on $X(k'_v)$. The space of global Schwartz function on $X(\mathbb{A}')$ is defined to be the restricted tensor product:

$$\mathcal{S}(X(\mathbb{A}')) = \bigotimes_v {}' \mathcal{S}(X(k'_v))$$

with respect to the functions $\{\mathbf{1}_{X(\mathfrak{o}_{k',v})} \mid v \notin S\}$. Suppose $f = \prod_v f_v \in \mathcal{S}(X(\mathbb{A}'))$ is a factorizable Schwartz function. For any finite set of places S, we define

$$f^S = \prod_{v \notin S} f_v, \quad f_S = \prod_{v \in S} f_v.$$

- We fix some measure on each group we integrate over. This choice is not essential since we only concern the non-vanishing problem.

2. Weil representations of the unitary groups

2.1. **Local theory.** In this section, we briefly review the Weil representations of the unitary groups. We are in the local situation as defined in Section 1.2. Let $(V, \langle -, - \rangle_V)$ and $(W, \langle -, - \rangle_W)$ be two Hermitian spaces of dimension n and m over k. The quadratic space

$$(\mathrm{Res}_{k/k'} V \otimes W, \ \widetilde{\mathrm{Tr}}(\langle -, - \rangle_V \otimes \langle -, - \rangle_W))$$

is a symplectic space of dimension $2mn$ over k'. Denote the symplectic group associated to it simply by $\mathrm{Sp}(\mathrm{Res}\, V \otimes W)$. There is a metaplectic cover of $\mathrm{Sp}(\mathrm{Res}\, V \otimes W)$, which sits in the central extension

$$1 \to \mathbb{C}^\times \to \mathrm{Mp}(\mathrm{Res}\, V \otimes W) \to \mathrm{Sp}(\mathrm{Res}\, V \otimes W) \to 1.$$

Let ω_ψ be the Weil representation of $\mathrm{Mp}(\mathrm{Res}\, V \otimes W)$. Let $\boldsymbol{\mu} = (\mu^n, \mu^m)$ be a pair of characters. By [HK-S1996, § 1, 2], the character $\boldsymbol{\mu}$ determines a splitting

$$
\begin{array}{ccc}
 & & \mathrm{Mp}(\mathrm{Res}\, V \otimes W) \\
 & \overset{\iota_{\boldsymbol{\mu}}}{\nearrow} & \downarrow \\
\mathrm{U}(V) \times \mathrm{U}(W) & \longrightarrow & \mathrm{Sp}(\mathrm{Res}\, V \otimes W).
\end{array}
$$

We thus get a Weil representation $\omega_{\psi, \boldsymbol{\mu}, V, W}$ of $\mathrm{U}(V) \times \mathrm{U}(W)$. It is not hard to see that $\overline{\omega_{\psi, \boldsymbol{\mu}, V, W}} \simeq \omega_{\psi^{-1}, \boldsymbol{\mu}^{-1}, V, W}$. We sometimes suppress the subscripts and write only ω for the Weil representation when there is no confusion on the characters and various spaces involved.

If V is a split Hermitian space, i.e. $V \simeq V' \oplus V'^\vee$, the spaces V' and $V'^\vee$ being totally isotropic, the Weil representation can be written in an explicit way. In this case, the dimension of V is even and we set $2r = \dim V$. We choose a basis $e_1, \cdots, e_r$ (resp. $e_r^\vee, \cdots, e_1^\vee$) of V' (resp. $V'^\vee$) so that $(e_i, e_j^\vee)_V = \delta_{ij}$. The set of vectors $e_1, \cdots, e_r, e_r^\vee, \cdots, e_1^\vee$ thus form a basis of V. The Hermitian form of V under this basis is thus $\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^\cdot} & \\ 1 & & \end{pmatrix}$. The decomposition $\mathrm{Res}\, V \otimes W = V' \otimes W \oplus V'^\vee \otimes W$ gives a complete polarization of the symplectic space $\mathrm{Res}\, V \otimes W$. The Weil representation can be realized on the space of Schwartz functions on $V'^\vee \otimes W$. Explicitly, identify $V'^\vee \otimes W$ with W^r, for $h \in \mathrm{U}(W)$ and $\phi \in \mathcal{S}(W^r)$, we have

$$
\begin{aligned}
\omega_{\psi, \boldsymbol{\mu}}(h)\phi(x) &= |\det h|^{-\frac{m}{2}} \mu(\det h)^n \phi(h^{-1}x), \\
\omega_{\psi, \boldsymbol{\mu}}(m(a))\phi(x) &= |\det a|^{\frac{n}{2}} \mu(\det a)^m \phi(xa), \\
\omega_{\psi, \boldsymbol{\mu}}(n(b))\phi(x) &= \psi(\mathrm{Tr}\, bT(x))\phi(x), \\
\omega_{\psi, \boldsymbol{\mu}}(w_r)\phi(x) &= \gamma_W \widehat{\phi}(x),
\end{aligned}
$$

(2.1.1)

where $a \in \mathrm{GL}_r(k)$, $b \in \mathrm{Mat}_{r,r}(k')$, $a^* = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^\cdot} & \\ 1 & & \end{pmatrix} {}^t a^{\tau, -1} \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^\cdot} & \\ 1 & & \end{pmatrix}$ and

$$m(a) = \begin{pmatrix} a & \\ & a^* \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad w_r = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^\cdot} & \\ 1 & & \end{pmatrix}$$

are elements in $\mathrm{U}(V)$. The constant γ_W is the Weil index and is an eighth root of unity. The Fourier transform is defined by

$$\widehat{\phi}(x) = \int_{W^r} \phi(y) \psi'\left(\frac{1}{2}\widetilde{\mathrm{Tr}}(x, y)_W\right) \mathrm{d}y,$$

where the measure $\mathrm{d}y$ is a self-dual additive measure on W^r. The matrix

$$T(x) = \frac{1}{2}((x_i, x_j)_W)_{1 \le i, j \le r}$$

is the moment matrix of x.

There is a natural nondegenerate pairing between ω and $\overline{\omega}$. When V is split, this pairing takes a particular simple form

$$\langle \phi, \phi' \rangle = \int\limits_{W^r} \phi(x)\phi'(x)\mathrm{d}x.$$

If $W \simeq E$ and the hermitian form is given by $\langle x, y \rangle = x^\tau y$, then we usually speak of the Weil representation of $\mathrm{U}(V)$, instead of the Weil representation of $\mathrm{U}(V) \times \mathrm{U}(W)$. This Weil representation is denote by $\omega_{\psi,\mu,V}$, or $\omega_{\psi,\mu}$ when there is no confusion about the space V.

Let π be an irreducible admissible representation of $\mathrm{U}(V) \times \mathrm{U}(V)$. A (local) Fourier–Jacobi model of π is an element in

$$\mathrm{Hom}_{\mathrm{U}(V)}\left(\pi \otimes \overline{\omega}, \mathbb{C}\right),$$

where ω is the Weil representation of $\mathrm{U}(V)$. The Fourier–Jacobi model is unique if exists. More precisely, we have

Theorem 2.1.1 (c.f. [Sun2012], [SZ2012]). *Let the notation be as above. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{U}(V)}\left(\pi \otimes \overline{\omega}, \mathbb{C}\right) \leq 1.$$

The local GGP conjecture predicts that in each generic Vogan packet, there is exactly one representation π with

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{U}(V)}\left(\pi \otimes \overline{\omega}, \mathbb{C}\right) = 1.$$

Moreover, this representation can be made explicit using ϵ-factors [GGP2012, Conjecture 17.3]. It is also checked in [GGP2012, § 18] that this conjecture does not depend on various choices of the characters. When k' is non-archimedean, Gan and Ichino [GI] has proved the local GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ in the non-archimedean case by reducing it to the local GGP conjecture for $\mathrm{U}(n+1) \times \mathrm{U}(n)$ which has been proved by Beuzart-Plessis [BP].

2.2. **Global theory.** Now we turn to the global theory of the Weil representation. We are in the global situation as defined in Section 1.2. Let $(V, \langle -, - \rangle_V)$ and $(W, \langle -, - \rangle_W)$ be two Hermitian spaces of dimension n and m over k.

By taking restricted tensor products of the local Weil representations, one obtains a global Weil representation $\omega_{\psi,\boldsymbol{\mu},V,W}$. This representation is realized on the space of Schwartz functions on a Lagrangian subspace $\mathbf{L} \subset \mathrm{Res}\, V \otimes W$. For different choices of the Lagrangian subspaces, the realizations are related by some (partial) Fourier transforms. If V is a split Hermitian space over k, then one has the similar explicit formulae for the global Weil representation as (2.1.1). As in the local situation, when $W \simeq E$ and the hermitian form is given by $\langle x, y \rangle = x^\tau y$, we speak of the Weil representation of the unitary group $\mathrm{U}(V)$ instead of the Weil representation of $\mathrm{U}(V) \times \mathrm{U}(W)$.

For any Schwartz function $\phi \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$, we define the theta series on $\mathrm{U}(V)(\mathbb{A}') \times \mathrm{U}(W)(\mathbb{A}')$ as

$$\theta_{\psi,\boldsymbol{\mu},V,W}(g,h,\phi) = \sum_{x \in \mathbf{L}(k')} \omega_{\psi,\boldsymbol{\mu},V,W}(g,h)\phi(x),$$

where $g \in \mathrm{U}(V)(\mathbb{A}')$ and $h \in \mathrm{U}(W)(\mathbb{A}')$. This is an automorphic form on $\mathrm{U}(V)(\mathbb{A}') \times \mathrm{U}(W)(\mathbb{A}')$. When $\dim W = 1$, we speak of the theta series on the unitary group $\mathrm{U}(V)$. This Weil representation is denoted by $\omega_{\psi,\mu,V}$, or $\omega_{\psi,\mu}$ when there is no confusion about the space V. We often suppress all the subscripts when there is no confusion about the characters and spaces involved.

Let π and σ be two irreducible cuspidal automorphic representations of $\mathrm{U}(V)(\mathbb{A}')$. Let $\varphi_\pi \in \pi$ and $\varphi_\sigma \in \sigma$. The integral

$$\int\limits_{\mathrm{U}(V)(k')\backslash \mathrm{U}(V)(\mathbb{A}')} \varphi_\pi(g)\varphi_\sigma(g)\theta_{\psi^{-1},\mu^{-1},V}(g,\phi)\mathrm{d}g$$

is called a Fourier–Jacobi period. It gives an element in

$$\mathrm{Hom}_{\mathrm{U}(V)(\mathbb{A}')}(\pi \otimes \sigma \otimes \overline{\omega}, \mathbb{C}).$$

3. Relative trace formula on the general linear group

3.1. **Orbits and orbital integrals.** In this section, we are in the local situation. Let V be a vector space over k of dimension n. By choosing a basis of V and the dual basis of $V^\vee$, the space V (resp. $V^\vee$) is identified with k^n (resp. k_n). The group $\mathrm{GL}(V)$ is identified with GL_n. Let $\mathsf{G} = \mathrm{Res}_{k/k'}(\mathrm{GL}_n \times \mathrm{GL}_n)$ be a reductive group over k' and $\mathsf{H}_1 = \mathrm{Res}_{k/k'}\mathrm{GL}_n$, $\mathsf{H}_2 = \mathrm{GL}_n \times \mathrm{GL}_n$ its subgroups. The group H_1 is embedded in G diagonally while H_2 is embedded in G componentwisely. The centers of these groups are denoted by Z, Z_1 and Z_2 respectively.

Recall that we have introduced the symmetric space S_n over k' with

$$\mathsf{S}_n(k') = \{x \in \mathrm{GL}_n(k) \mid xx^\tau = 1\}.$$

Moreover, there is a surjective map

$$\sigma : \mathrm{GL}_n(k) \to \mathsf{S}_n(k'), \quad \gamma \mapsto \gamma\gamma^{-1,\tau}.$$

We introduce an integral transform which is also denoted by σ

$$\sigma : \mathcal{S}(\mathrm{GL}_n(k)) \to \mathcal{S}(\mathsf{S}_n(k')), \quad \sigma(F)(\gamma) = \int_{\mathrm{GL}_n(k')} F(\gamma h)\mathrm{d}h.$$

If F is compactly supported, so is $\sigma(F)$. The map σ is surjective by Dixmier–Malliavin theorem [DM1978].

Define a symmetric space $\mathsf{X} = \mathsf{S}_n \times k'_n \times k'^n$ over k' and $\mathrm{GL}_n(k')$ acts on $\mathsf{X}(k')$ from the right by

$$[\xi, v^\vee, v].g = [g^{-1}\xi g, v^\vee g, g^{-1}v].$$

Sometimes it is also convenient to consider the right action of $\mathsf{H}_1(k') \times \mathsf{H}_2(k')$ on $\mathsf{G}(k') \times k'_n \times k'^n$ by

$$[\gamma, v^\vee, v].(g, h) = [g^{-1}\gamma h, v^\vee h_1, h_1^{-1}v], \quad h = (h_1, h_2) \in \mathrm{GL}_n(k') \times \mathrm{GL}_n(k').$$

It is straight forward to check that an element $[\gamma = (\gamma_1, \gamma_2), v^\vee, v]$ is regular semisimple if and only if $[\sigma(\gamma_1^{-1}\gamma_2), v^\vee, v]$ is regular semisimple.

Let us introduce a (partial) Fourier transform. First the identifications $V \simeq k^n$ and $V^\vee \simeq k_n$ give an action of $\mathrm{Gal}(k/k')$ on V and $V^\vee$. Thus we have a decomposition (as k' vector spaces)

$$V = V^+ \oplus V^-, \qquad V^\vee = V^{\vee,+} \oplus V^{\vee,-},$$

and we define $V^\dagger = V^+ \oplus V^{\vee,+}$. The superscripts $\pm$ indicate the sign of the eigenvalue τ on each space. Let $\Phi \in \mathcal{S}(V^\vee)$ be a Schwartz function on $V^\vee$. Define its (partial) Fourier transform by

$$\Phi^\dagger(v^{\vee,+}, v^+) = \int_{V^{\vee,-}} \Phi(v^{\vee,+}, v^{\vee,-})\psi(\langle v^{\vee,-}, v^+\rangle)\mathrm{d}v^{\vee,-}.$$

The pairing is the canonical pairing between V and $V^\vee$. The measure is chosen to be the self-dual measure on $V^{\vee,-}$ with respect to ψ. By our choice of the basis, the space $V^\dagger$ is identified with $k'_n \times k'^n$. Let ω be the Weil representation of $\mathrm{GL}_n(k)$, realized on $\mathcal{S}(k_n)$ by

$$\omega(g)\phi(x) = \mu(\det g)|\det g|^{\frac{1}{2}}\phi(xg).$$

Then there is a unique action $\omega^\dagger$ of $\mathrm{GL}_n(k)$ on $\mathcal{S}(k'_n \times k'^n)$ which makes the above defined Fourier transform equivariant. When $h \in \mathrm{GL}_n(k')$, $\omega^\dagger$ takes a particularly simple form

$$\omega^\dagger(h)\Phi^\dagger(v^{\vee,+}, v^+) = \eta(\det h)|\det h|^{\frac{1}{2}}\Phi^\dagger(v^{\vee,+}h, h^{-1}v^+).$$

One should note that the absolute value here is the absolute value of k even though $h \in \mathrm{GL}_n(k')$.

For a regular semisimple element $[\gamma, v^\vee, v] \in \mathsf{G}(k') \times k'_n \times k'^n$ and a test function (F, Φ) where $F \in \mathcal{C}_c^\infty(\mathsf{G}(k'))$ and $\Phi \in \mathcal{S}(k_n)$, we define the orbital integral

(3.1.1)
$$\mathbf{O}^{\psi,\mu}(s, [\gamma, v^\vee, v], F, \Phi) = \int_{\mathsf{H}_2(k')}\int_{\mathsf{H}_1(k')} F(g^{-1}\gamma h)\mu^{-1}(\det g)$$
$$\left(\omega(h_1^{-1}\gamma_1^{-1}g)\Phi\right)^\dagger(v^\vee h_1, h_1^{-1}v)|\det gh_1^{-1}|^{s-\frac{1}{2}}\mathrm{d}g\mathrm{d}h.$$

Up to a sign, this orbital integral depends only on the orbit. One should note that this orbital integral depends on the choice of the characters ψ and μ. We write simply $\mathbf{O}^{\psi,\mu}([\gamma, v^\vee, v], F, \Phi)$ for the orbital

integral at $s = \frac{1}{2}$. We treat only this case in the following as it is the case we are interested in. The orbital integral extends by linearity to all the test functions in $\mathcal{C}_c^\infty(\mathsf{G}(k')) \otimes \mathcal{S}(k_n)$.

The orbital integral can be simplified as follows. Each orbit has a representative of the form $[(1, \gamma), v^\vee, v]$. Writing $h = (h_1, h_2)$ and making a change of variable $g \mapsto h_1 g$, one has

$$(3.1.1) = \iint_{(\mathrm{GL}_n(k'))^2} \int_{\mathsf{H}_1(k')} F(g^{-1}, g^{-1}h_1^{-1}\gamma h_2)\mu^{-1}(\det g)\eta(\det h_1)(\omega(g)\Phi)^\dagger(v^{\vee,+}h_1, h_1^{-1}v^+)\mathrm{d}g\mathrm{d}h_1\mathrm{d}h_2,$$

with $\gamma \in \mathrm{GL}_n(k)$ and $(v^{\vee,+}, v^+) \in k_n' \times k'^n$.

For any test function (F, Φ), we define a function $\Upsilon_{F,\Phi}$ on $\mathsf{X}(k')$ as

$$(3.1.2) \qquad \Upsilon_{F,\Phi}(\sigma(\gamma), v^\vee, v) = \int_{\mathrm{GL}_n(k')} \int_{\mathsf{H}_1(k')} F(g^{-1}, g^{-1}\gamma h_2)\mu^{-1}(\det g)(\omega(g)\Phi)^\dagger(v^{\vee,+}, v^+)\mathrm{d}g\mathrm{d}h_2.$$

The local orbital integral is thus simplified to an orbital integral on X,

$$(3.1.3) \qquad \mathbf{O}^{\psi,\mu}([\xi, v^\vee, v], \Upsilon) = \int_{\mathrm{GL}_n(k')} \Upsilon([\xi, v^\vee, v].h)\eta(\det h)\mathrm{d}h.$$

We shall refer to the integral $(3.1.3)$ as the simplified orbital integral on the general linear group. We shall drop the superscripts when there is no confusion with the characters.

Lemma 3.1.1. *For any test function (F, Φ) and any regular semisimple orbit $[\gamma, v^\vee, v]$, the orbital integral $(3.1.1)$ is absolutely convergent.*

Proof. We first note that since F and Φ are both Schwartz functions, the integral transform $\Upsilon_{F,\Phi}$ is again a Schwartz function on $\mathsf{X}(k')$.

The orbit $[\gamma, v^\vee, v]$ is regular semisimple if and only if $[\sigma(\gamma_1^{-1}\gamma_2), v^\vee, v] \in \mathsf{X}(k')$ is regular semisimple. The orbit $[\sigma(\gamma_1^{-1}\gamma_2), v^\vee, v]$ is thus closed in $\mathsf{X}(k')$. The function $\Upsilon_{F,\Phi}$ restricted to this orbit is again a Schwartz function. Therefore the integral is absolutely convergent. $\square$

The following lemma will be used in the proof of the existence the smooth transfer.

Lemma 3.1.2. *If k' is non-archimedean, then the integral transform $(3.1.2)$*

$$\mathcal{S}(\mathsf{G}(k')) \otimes \mathcal{S}(k_n) \to \mathcal{S}(\mathsf{X}(k')), \quad F \otimes \Phi \mapsto \Upsilon_{F,\Phi}$$

is surjective.

Proof. Let $F_0 \otimes \Phi_0 \in \mathcal{S}(\mathsf{S}_n(k')) \otimes \mathcal{S}(k_n' \times k'^n)$. Choose an $\widetilde{F_0} \in \mathcal{S}(\mathrm{GL}_n(k))$ such that

$$\int_{\mathrm{GL}_n(k')} \widetilde{F_0}(\gamma h_2)\mathrm{d}h_2 = F_0(\sigma(\gamma)),$$

for any $\gamma \in \mathrm{GL}_n(k)$. This is clearly possible. For instance, we can take

$$\widetilde{F_0}(\gamma) = \frac{1}{\mathrm{vol}\,\Omega} F_0(\sigma(\gamma))\,\mathbf{1}_\Omega(\gamma)$$

for any open compact subset Ω of $\mathrm{GL}_n(k)$. Let $\Phi \in \mathcal{S}(k_n)$ be the inverse Fourier transform of Φ_0, i.e. $\Phi^\dagger = \Phi_0$. Let $U \subset \mathrm{GL}_n(k)$ be an open compact neighborhood of identity matrix, such that for any $g \in U$, $\gamma \in \mathrm{GL}_n(k)$, we have

$$\widetilde{F_0}(g^{-1}\gamma) = \widetilde{F_0}(\gamma), \quad \omega(g)\Phi = \Phi.$$

This is possible since $\widetilde{F_0}$ and Φ are locally constant and compactly supported. Let $F_1(g) = \mathbf{1}_U(g)\cdot\mu^{-1}(\det g)$, $F_2(g) = (\mathrm{vol}\,U)^{-1}\cdot\widetilde{F_0}(g)$, then

$$\int_{\mathrm{GL}_n(k')} \int_{\mathsf{H}_1(k')} F_1(g^{-1})F_2(g^{-1}\gamma h_2)\mu^{-1}(\det g)\,(\omega(g)\Phi)^\dagger\,(v^{\vee,+}, v^+)\mathrm{d}g\mathrm{d}h_2 = F_0(\gamma)\Phi_0(v^{\vee,+}, v^+).$$

This shows that $F_1 \otimes F_2 \otimes \Phi$ maps to $F_0 \otimes \Phi_0$. $\square$

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3.2. The distribution. In this section, we are always in the global situation as defined in Section 1.2.

In order to define the distribution, let us recall the Rankin–Selberg convolution of $\mathrm{GL}_n \times \mathrm{GL}_n$. This is the Fourier–Jacobi model on the general linear group.

For any Schwartz function $\Phi \in \mathcal{S}(\mathbb{A}_n)$, define the theta series

$$\Theta_\mu(s, g, \Phi) = |\det g|^{s - \frac{1}{2}} \sum_{v \in k_n} \omega(g)\Phi(v).$$

Let Π be an irreducible cuspidal automorphic representations of $\mathsf{G}(\mathbb{A}')$ with central character ω_Π. For simplicity, we assume that ω_Π is unitary and distinguished. In other words, $\omega_\Pi|_{\mathbb{A}'^\times \times \mathbb{A}'^\times}$ is trivial. This in particular implies that $\omega_\Pi \cdot \mu^{-1} \neq 1$. The Rankin–Selberg integral is defined to be

$$(3.2.1) \qquad Z(s, \varphi_\Pi, \Phi) = \int\limits_{\mathrm{GL}_n(k) \backslash \mathrm{GL}_n(\mathbb{A})} \varphi_\Pi(g) \Theta_{\mu^{-1}}(s, g, \Phi) \mathrm{d}g,$$

where $\varphi_\Pi \in \Pi$ and Φ is a Schwartz function on $\mathbb{A}_n$. This integral is absolutely convergent when $\operatorname{Re} s \gg 0$, and has a meromorphic continuation to the entire complex plane.

Remark 3.2.1. The Rankin–Selberg integral is usually defined to be

$$\int\limits_{\mathrm{GL}_n(k) \backslash \mathrm{GL}_n(\mathbb{A})} \varphi_\Pi(g) \Theta^*_{\mu^{-1}}(s, g, \Phi) \mathrm{d}g,$$

where Θ^* is the same as Θ, except that the sum is over $k_n - \{0\}$. Because of our assumption on Π, there is no difference using Θ or Θ^*.

The following celebrated theorem of Jacquet–Piatetski-Shapiro–Shalika [JPSS1983] can be viewed as the global Gan–Gross–Prasad conjecture in the case $k = k' \times k'$.

Theorem 3.2.2 ([JPSS1983])**.** *Write* $\Pi = \Pi_1 \boxtimes \Pi_2$. *There is a choice of data* $\varphi_1 \in \Pi_1$, $\varphi_2 \in \Pi_2$ *and* Φ *such that the Rankin–Selberg integral* (3.2.1) *does not vanish at* $s = \frac{1}{2}$ *if and only if* $L(\frac{1}{2}, \Pi_1 \times \Pi_2 \otimes \mu^{-1}) \neq 0$.

For $F \in \mathcal{C}_c^\infty(\mathsf{G}(\mathbb{A}'))$, we define the kernel functions

$$\widetilde{K}_F(g, h) = \sum_{\gamma \in \mathsf{G}(k')} F(g^{-1}\gamma h),$$

and

$$K_F(g, h) = \int\limits_{\mathsf{Z}_2(k') \backslash \mathsf{Z}_2(\mathbb{A}')} \widetilde{K}_F(g, zh) \mathrm{d}z.$$

Suppose n is odd. For the test function (F, Φ), we define

$$\mathscr{I}(s, F, \Phi) = \int\limits_{\mathsf{Z}_2(\mathbb{A}')\mathsf{H}_2(k') \backslash \mathsf{H}_2(\mathbb{A}')} \int\limits_{\mathsf{H}_1(k') \backslash \mathsf{H}_1(\mathbb{A}')} K_F(g, h) \Theta^*_{\mu^{-1}}(s, g, \Phi) \, \mathrm{d}g \mathrm{d}h$$

$$(3.2.2)$$

$$= \int\limits_{\mathsf{H}_2(k') \backslash \mathsf{H}_2(\mathbb{A}')} \int\limits_{\mathsf{H}_1(k') \backslash \mathsf{H}_1(\mathbb{A}')} \widetilde{K}_F(g, h) \Theta^*_{\mu^{-1}}(s, g, \Phi) \, \mathrm{d}g \mathrm{d}h.$$

If n is even, we should consider the integral

$$\mathscr{I}(s, F, \Phi) = \int\limits_{\mathsf{H}_2(k') \backslash \mathsf{H}_2(\mathbb{A}')} \int\limits_{\mathsf{H}_1(k') \backslash \mathsf{H}_1(\mathbb{A}')} \widetilde{K}_F(g, h) \Theta^*_{\mu^{-1}}(s, g, \Phi) \, \eta(\det h_1 h_2) \mathrm{d}g \mathrm{d}h.$$

Both cases where n is odd or even will lead to the same transfer problem and the fundamental lemma. In the following, we shall focus on the case where n is odd and leave the minor modification for n even to the reader.

When $s = 1/2$, we suppress all s and simply write $\mathscr{I}(F, \Phi) = \mathscr{I}(1/2, F, \Phi)$. In general, these integrals might not converge. We say a test function (F, Φ) is *good* if the following conditions hold.

(1) At a nonarchimedean split place v_1 of k', F_{v_1} is a truncated matrix coefficient of a supercuspidal representation. This means
$$F_{v_1}(g) = \widetilde{F_{v_1}}(g)\,\mathbf{1}_{\mathsf{G}^*(k'_{v_1})}(g).$$
Here $\widetilde{F_{v_1}}$ is a matrix coefficient of a supercudpidal representation of $\mathsf{G}(k'_{v_1})$ and
$$\mathsf{G}^*(k'_{v_1}) = \{(g_1, g_2) \in \mathsf{G}(k'_{v_1}) \mid |\det g_1| = |\det g_2| = 1\}.$$
In particular, F_{v_1} is compactly supported.

(2) At another nonarchimedean split place $v_2 \neq v_1$, the function $\Upsilon_{F_{v_1}, \Phi_{v_1}}$ is supported on the regular semisimple locus of $\mathsf{X}(k'_{v_2})$.

(3) For all archimedean places v, $k_v = k'_v \times k'_v$, Φ is a finite linear combination of functions of the form $\Phi \otimes \Phi'$ where $\Phi, \Phi' \in \mathcal{S}(k'_{v,n})$. Moreover, F_v is K_v-finite where K_v is a maximal compact subgroup of $\mathsf{G}(k'_v)$.

Proposition 3.2.3. *For good test functions (F, Φ), the integral above defining the distributions is absolutely convergent. Moreover, if $F = \prod_v F_v$ and $\Phi = \prod_v \Phi_v$ are both factorizable, then*
$$\mathscr{I}(F, \Phi) = \sum_{[\gamma, v^\vee]} \prod_v \mathbf{O}_v^{\psi, \mu}([\gamma, v^\vee], F_v, \Phi_v),$$
where the right hand side is a sum over regular semisimple orbits and $\mathbf{O}_v^{\psi, \mu}$ is the local orbital integral defined in Section 3.1.

Proof. The distribution can be written as
$$(3.2.2) = \int\limits_{\mathsf{H}_2(k')\backslash \mathsf{H}_2(\mathbb{A}')} \int\limits_{\mathsf{H}_1(k')\backslash \mathsf{H}_1(\mathbb{A}')} \sum_{\gamma \in \mathsf{G}(k')} \sum_{v \in k_n} F(g^{-1}\gamma h)\Phi(vg)\mu^{-1}(\det g)\mathrm{d}g\mathrm{d}h.$$
By Poisson summation formula,
$$\sum_{v \in k_n} \Phi(vg) = \sum_{v \in k_n} \Phi(vh_1(h_1^{-1}\gamma_1^{-1}g)) = \sum_{\substack{v^{\vee,+} \in k'_n \\ v^+ \in k'^n}} |\det gh_1^{-1}|^{-\frac{1}{2}} (\omega(h_1^{-1}\gamma_1^{-1}g)\Phi)^\dagger(v^{\vee,+}h_1, h_1^{-1}v^+).$$
Since at the place v_2, the test function is chosen to be supported on the regular semisimple locus,,
$$(3.2.2) = \sum \iint\limits_{(\mathrm{GL}_n(\mathbb{A}'))^2} \int\limits_{\mathsf{H}_1(\mathbb{A}')} F(g^{-1}\gamma h)(\omega(h_1^{-1}\gamma_1^{-1}g)\Phi)^\dagger(v^{\vee,+}h_1, h_1^{-1}v^+)$$
$$\mu^{-1}(\det g)\eta(\det h_1)\mathrm{d}g\mathrm{d}h_1\mathrm{d}h_2,$$
where sum is over regular semisimple orbits.

The convergence of the global orbital integral can be proved in the same way as Lemma 3.1.1.

We now show that the outer sum is convergent. Making a change of variables $g \mapsto h_1^{-1}\gamma_1 g$, we obtain that the orbital integral becomes
$$\iint\limits_{(\mathrm{GL}_n(\mathbb{A}'))^2} \int\limits_{\mathrm{GL}_n(\mathbb{A})} F(g^{-1}, g^{-1}h_1^{-1}\gamma_1^{-1}\gamma_2 h_2)(\omega(g)\Phi)^\dagger(v^{\vee,+}h_1, h_1^{-1}v^+)\mathrm{d}g\mathrm{d}h_1\mathrm{d}h_2.$$
Integrating over g and h_2 first, one sees that the orbital integral becomes
$$\int\limits_{\mathrm{GL}_n(\mathbb{A}')} \widetilde{F}(h_1^{-1}\gamma h_1)\Phi_1(v^{\vee,+}h_1, h_1^{-1}v^+)\mathrm{d}h_1,$$
where $\gamma = \sigma(\gamma_1^{-1}\gamma_2)$, Φ_1 is some Schwartz function on $\mathbb{A}'_n \times \mathbb{A}'^n$ and $\widetilde{F}$ is some compactly supported function on $\mathsf{S}_n(\mathbb{A}')$. The orbit $[\gamma, v^{\vee,+}, v^+] \in \mathsf{S}_n(k') \times k'_n \times k'^n$ is regular semisimple. Since the support of $\widetilde{F}$ is compact, we see that there are only finitely many possible regular semisimple $\gamma \in \mathsf{S}_n(k') /\!\!/ \mathrm{GL}_n(k')$ such that $\widetilde{F}(h^{-1}\gamma h) = 0$ for some h, where GL_n acts on S_n by conjugation. Therefore to show that the sum is convergent, we only need to show that for a fixed regular semisimple γ, the summation over regular

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semisimple $v^{\vee,+}$ and v^+, modulo the action of the stablizer of γ, is convergent. Let Stab_γ be the stabilizer of γ in GL_n. It is a reductive group over k'. Let K_γ be a subset of $\mathrm{GL}_n(\mathbb{A}')$ such that

$$\mathrm{GL}_n(\mathbb{A}') = K_\gamma \cdot \mathrm{Stab}_\gamma(\mathbb{A}').$$

The orbital integral can be written as

$$\int\limits_{K_\gamma} \int\limits_{\mathrm{Stab}_\gamma(\mathbb{A}')} \widetilde{F}(k^{-1}\gamma k)\Phi_1(v^{\vee,+}kz, z^{-1}k^{-1}v^+)\mathrm{d}z\mathrm{d}k.$$

As the support of $\widetilde{F}$ is compact, the integration with respect to k is in fact over some compact region. The function

$$\Phi_2(v^{\vee,+}, v^+) = \int\limits_{K_\gamma} \widetilde{F}(k^{-1}\gamma k)\Phi_1(v^{\vee,+}k, k^{-1}v^+)\mathrm{d}k$$

is then a Schwartz function in $v^{\vee,+}, v^+$.

Therefore we are reduced to show that sum

(3.2.3)
$$\sum \int\limits_{\mathrm{Stab}_\gamma(\mathbb{A}')} \Phi_2(v^{\vee,+}z, z^{-1}v^+)\mathrm{d}z$$

is absolutely convergent for some Schwartz function Φ_2. Here the summation is over the set

$$\left\{[v^{\vee,+}, v^+] \in k'_n \times k'^n \mid \det(v^{\vee,+}\gamma^{i+j}v^+)_{i,j=0}^{n-1} \neq 0\right\} /\!\!/ \mathrm{Stab}_\gamma(k').$$

The group Stab_γ acts on $k'_n \times k'^n$ by

$$z.[v^{\vee,+}, v^+] = [v^{\vee,+}z, z^{-1}v^+],$$

and the point $[v^{\vee,+}, v^+]$ is regular semisimple precisely when $\det(v^{\vee,+}\gamma^{i+j}v^+)_{i,j=0}^{n-1} \neq 0$. The regular semisimple elements have trivial stabilizer. Let $X = k'_n \times k'^n$, viewed as an algebraic variety over k' and $Q = X /\!\!/ \mathrm{Stab}_\gamma$ the categorical quotient. Then Q is isomorphic to $\mathbf{A}_n$, the affine n-space over k', and the isomorphism is given by

$$[v^{\vee,+}, v^+] \mapsto (v^{\vee,+}\gamma^i v^+)_{i=0,\cdots,n-1}.$$

The regular semisimple orbits form an Zariski open dense subset Q_{rss} of Q, and the complement is denoted by Z. The summation in (3.2.3) is thus over $Q_{\mathrm{rss}}(k')$.

Let V be the subset of $X(\mathbb{A}')$ defined by

$$\left\{[v^{\vee,+}, v^+] \mid \left|\det(v^{\vee,+}\gamma^{i+j}v^+)_{i,j=0}^{n-1}\right|_{\mathbb{A}'} \leq \frac{1}{2}\right\}.$$

This is a closed subset of $\mathbb{A}'_n \times \mathbb{A}'^n$ and is stable under the action of Stab_γ. It follows from the product formula that there is no regular semisimple element in $X(k') \cap V$. Let f_V be the characteristic function of the set $X(\mathbb{A}')\backslash V$. Then one sees that by replacing Φ_2 by $\Phi_2 f_V$, we may assume that the support of Φ_2 is contained in $X(\mathbb{A}')\backslash V$. Without loss of generality, one may further assume that $\Phi_2 = \prod_w \Phi_{2,w}$ is factorizable. We use w to denote a place of k' in order not to raise any confusion with the variable v. We can now replaces the outer sum over the regular semisimple orbits by the sum over all the rational points of Q.

The integral

$$\int\limits_{\mathrm{Stab}_\gamma(\mathbb{A}'_\mathrm{f})} \Phi_{2,\mathrm{f}}(v^{\vee,+}z, z^{-1}v^+)\mathrm{d}z$$

gives a compactly supported function on $Q(\mathbb{A}'_\mathrm{f})$. Furthermore, the support is contained in $Q_{\mathrm{rss}}(\mathbb{A}'_\mathrm{f})$. Therefore there is a constant A such that any element $[v^{\vee,+}, v^+]$ in the support satisfies

$$|\det(v^{\vee,+}\gamma^{i+j}v^+)_{i,j=0}^{n-1}|_{\mathbb{A}'_\mathrm{f}} \leq A.$$

This implies that if $[v^{\vee,+}, v^+]$ in the support of Φ, then

(3.2.4)
$$\prod_{w|\infty} |\det(v^{\vee,+}\gamma^{i+j}v^+)_{i,j=0}^{n-1}|_w \geq \frac{1}{2A}.$$

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Let w be an infinite place of k'. We now consider the integral

$$\int_{\mathrm{Stab}_\gamma(k'_w)} \Phi_{2,w}(v^{\vee,+}z, z^{-1}v^+)\mathrm{d}z.$$

It is clear that this gives a function on $Q(k'_w)$ supported on the regular semisimple locus. Since $\Phi_{2,w}$ is a Schwartz function, for any positive real numbers α and β, there ia constant $C_{\alpha\beta}$, such that

$$|\Phi_{2,w}(v^{\vee,+}z, z^{-1}v^+)| \le C_{\alpha\beta} \cdot \frac{1}{\left(1 + \sum_{i=0}^{n-1}|v^{\vee,+}\gamma^i z|_w + \sum_{i=0}^{n-1}|z^{-1}\gamma^{-i}v^+|_w\right)^\alpha}$$

$$\frac{1}{(1 + |v^{\vee,+}v^+|_w + \cdots + |v^{\vee,+}\gamma^{n-1}v^+|_w)^\beta}.$$

Here we have used the fact that $z\gamma = \gamma z$. For sufficiently large α, there is a constant C'_α, such that

$$\int_{\mathrm{Stab}_\gamma(k'_w)} \frac{1}{\left(1 + \sum_{i=0}^{n-1}|v^{\vee,+}\gamma^i z|_w + \sum_{i=0}^{n-1}|z^{-1}\gamma^{-i}v^+|_w\right)^\alpha}\mathrm{d}z \le C'_\alpha \left|\det(v^{\vee,+}\gamma^{i+j}v^+)_{i,j=0}^{n-1}\right|_w^{-1}.$$

We conclude that there is a constant C_β, such that

$$\int_{z\in\mathrm{Stab}_\gamma(k'_w)} |\Phi_{2,w}(v^{\vee,+}z, z^{-1}v^+)|\mathrm{d}z$$

$$\le C_\beta \times \left|\det(v^{\vee,+}\gamma^{i+j}v^+)_{i,j=0}^{n-1}\right|_w^{-1} \times \frac{1}{(1 + |v^{\vee,+}v^+|_w + \cdots + |v^{\vee,+}\gamma^{n-1}v^+|_w)^\beta}.$$

Thus for some constant B depending on β,

$$\left|\prod_{w|\infty} \int_{\mathrm{Stab}_\gamma(k'_w)} \Phi_{2,w}(v^{\vee,+}z, z^{-1}v^+)\mathrm{d}z\right| \le B \times \frac{1}{(1 + |v^{\vee,+}v^+|_{\mathbb{A}'_\infty} + \cdots + |v^{\vee,+}\gamma^{n-1}v^+|_{\mathbb{A}'_\infty})^\beta}.$$

Here we have used the inequality (3.2.4). This shows that,

$$\left|P(v^{\vee,+}v^+, \cdots, v^{\vee,+}\gamma^{n-1}v^+) \times \prod_{w|\infty} \int_{\mathrm{Stab}_\gamma(k'_w)} \Phi_{2,w}(v^{\vee,+}z, z^{-1}v^+)\mathrm{d}z\right|,$$

as a function on $Q(\mathbb{A}'_\infty)$ is bounded where P is any polynomial function on $Q(\mathbb{A}'_\infty)$. Now the convergence of the sum (3.2.3) follows from the following well-known fact.

Suppose V is an n-dimensional vector space over k'. If $\Phi = \prod_w \Phi_w$ is a continuous function on $V(\mathbb{A}')$ with the properties that $\prod_{w\nmid\infty} \Phi_w$ is compactly supported and that $P \times \prod_{w|\infty} \Phi_w$ is bounded for any polynomial function P on $V(\mathbb{A}'_\infty)$, then $\sum_{v\in V(k')} \Phi(v)$ is convergent. $\qquad\square$

4. Relative trace formula on the unitary groups

4.1. **Orbits and orbital integrals.** In this section, we are in the local situation as defined in Section 1.2. Let $(V, \langle -, - \rangle)$ be a Hermitian space of dimension n over k. Let $G(V)$ be the reductive group $\mathrm{U}(V) \times \mathrm{U}(V)$, $H(V) = \mathrm{U}(V)$ which is embedded in $G(V)$ diagonally. By abuse of notation, we also write $\mathrm{U}(V)$ (resp. $G(V)$, $H(V)$) for its group of k' points when there is no confusion.

Let $\mathsf{Y} = \mathrm{U}(V) \times V^\vee$ be a symmetric space and $H(V)$ acts on it from the right by

$$[\xi, v^\vee].h = [h^{-1}\xi h, v^\vee h].$$

We write Y^V to indicate the Hermitian space V when it is needed.

Sometimes it is also convenient to consider the right action of $H(V)(k') \times H(V)(k')$ on $G(V)(k') \times V^\vee$

$$[\zeta, w^\vee].(h_1, h_2) = [h_1^{-1}\zeta h_2, w^\vee h_2].$$

An element $[\zeta = (\zeta_1, \zeta_2), w] \in G(V)(k') \times V^\vee$ is regular semisimple if and only if $[\zeta_1^{-1}\zeta_2, w] \in \mathsf{Y}$ is regular semisimple. Note that in each orbit, there is a representative with $\zeta_1 = 1$.

Let $\omega_{\psi,\mu}$ be the Weil representation of $U(V)(k')$. We shall drop the subscripts ψ and μ when there is no confusion about the characters. Let $V = \mathbf{L} + \mathbf{L}^\vee$ be a polarization. The Weil representation of $U(V)$ is realized on the space $\mathcal{S}(\mathbf{L})$. Define the (partial) Fourier transform

$$-^\ddagger : \mathcal{S}(\mathbf{L})^{\otimes 2} \to \mathcal{S}(V^\vee)$$

by

$$(4.1.1) \qquad (\phi_1 \otimes \phi_2)^\ddagger(w^\vee) = \int_{\mathbf{L}} \phi_1(x+z)\phi_2(x-z)\psi(\langle z, y\rangle)\mathrm{d}z.$$

The pairing is the one between $\mathbf{L}$ and $\mathbf{L}^\vee$, and $w^\vee = (x,y)$ where $x \in \mathbf{L}$ and $y \in \mathbf{L}^\vee$. The group $G(V)$ acts on $\mathcal{S}(\mathbf{L})^{\otimes 2}$ via $\omega_{\psi,\mu} \otimes \omega_{\psi^{-1},\mu^{-1}}$. There is a unique action $\omega^\ddagger$ on $\mathcal{S}(V^\vee)$ making the above Fourier transform equivariant. The action of an element of $H(V)$ takes a particular simple form, i.e. the right translation, and we have

$$(\omega_{\psi,\mu}(h)\phi_1 \otimes \omega_{\psi^{-1},\mu^{-1}}(h)\phi_2)^\ddagger(w) = (\phi_1 \otimes \phi_2)^\ddagger(wh).$$

For $[\zeta, w^\vee] = [(\zeta_1, \zeta_2), w^\vee] \in G(V)(k') \times V^\vee$, and a test function $(f, \phi_1 \otimes \phi_2)$ where $f \in \mathcal{C}_c^\infty(G(V)(k')$ and $\phi_1, \phi_2 \in \mathcal{S}(\mathbf{L}(k'))$, define the orbital integral

$$(4.1.2) \qquad \mathbf{O}^{\psi,\mu}([\zeta, w^\vee], f, \phi_1 \otimes \phi_2) = \iint_{(U(V)(k'))^2} f(g_1^{-1}\zeta g_2)\left(\omega_{\psi^{-1},\mu^{-1}}(g_2^{-1}\zeta_1^{-1}g_1)\phi_1 \otimes \phi_2\right)^\ddagger(w^\vee g_2)\mathrm{d}g_1\mathrm{d}g_2.$$

The integral $(4.1.2)$ depends only on the orbit, not a specific representative.

The orbital integral can be simplified along the same line as the general linear group side. Each orbit has an representative that is of the form $[(1, \zeta), w^\vee]$. Suppose $f = f_1 \otimes f_2$. Making a change of variable $g_1 \mapsto g_2 g_1$, one obtains

$$(4.1.2) = \iint_{(U(V)(k'))^2} f_1(g_1^{-1})f_2(g_1^{-1}g_2^{-1}\zeta g_2)\left(\omega_{\psi^{-1},\mu^{-1}}(g_1)\phi_1 \otimes \phi_2\right)^\ddagger(w^\vee g_2)\mathrm{d}g_1\mathrm{d}g_2.$$

For the test function $(f, \phi_1 \otimes \phi_2)$, we define a function on $U(V) \times V^\vee$ by

$$(4.1.3) \qquad \Psi_{f,\phi_1 \otimes \phi_2}([\zeta, w^\vee]) = \int_{U(V)(k')} f_1(g_1^{-1})f_2(g_1^{-1}\zeta)\left(\omega_{\psi^{-1},\mu^{-1}}(g_1)\phi_1 \otimes \phi_2\right)^\ddagger(w^\vee)\mathrm{d}g_1.$$

This is a Schwartz function on $U(V) \times V^\vee$. We define by linearity the function $\Psi_{f,\phi}$ for any $\phi = \sum_{j=1}^r \phi_{1,j} \otimes \phi_{2,j} \in \mathcal{S}(\mathbf{L})^{\otimes 2}$.

The orbital integral is thus simplified to

$$(4.1.4) \qquad \mathbf{O}^{\psi,\mu}(\Psi, [\xi, w^\vee]) = \int_{U(V)(k')} \Psi([\xi, w^\vee].h)\mathrm{d}h.$$

We shall refer to the integral $(4.1.4)$ as the simplified orbital integral on the unitary group. We shall drop the superscripts when there is no confusion with the characters.

Lemma 4.1.1. *If the orbit $[\zeta, w^\vee]$ is regular semisimple, then the integral $(4.1.2)$ is absolutely convergent.*

Proof. The proof is similar to the proof of Lemma 3.1.1. $\qquad\square$

The following lemma will be used in the proof of the existence of smooth transfer.

Lemma 4.1.2. *If k' is non-archimedean, then the integral transformation $(4.1.3)$*

$$\mathcal{S}(G(V)(k')) \otimes \mathcal{S}(\mathbf{L}(k') \times \mathbf{L}(k')) \to \mathcal{S}(\mathsf{Y}(k')), \quad (f, \phi) \mapsto \Psi_{f,\phi}$$

is surjective.

Proof. The proof is similar to the proof of Lemma 3.1.2. $\qquad\square$

4.2. The distribution. In this section, we are in the global situation as defined in Section 1.2. So let $(V, \langle -, - \rangle_V)$ be a Hermitian space over k, and $\mathrm{Res}\, V$ be the symplectic space over k' with the underline vector space V and the pairing

$$\mathrm{Tr}(j\langle -, - \rangle).$$

Let $\mathbf{L}$ be a Lagrangian of $\mathrm{Res}\, V$.

For $f \in \mathcal{C}_c^\infty(G(V)(\mathbb{A}'))$, define the kernel function

$$K_f(g, h) = \sum_{\zeta \in G(V)(k')} f(g^{-1}\zeta h).$$

For the test function $(f, \phi_1 \otimes \phi_2)$ where $f \in \mathcal{S}(G(V)(\mathbb{A}'))$ and $\phi_1, \phi_2 \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$, define the the distribution

$$(4.2.1) \qquad \mathscr{I}(f, \phi_1 \otimes \phi_2) = \iint\limits_{(\mathrm{U}(V)(k') \backslash \mathrm{U}(V)(\mathbb{A}'))^2} K_f(g, h) \theta_{\psi^{-1}, \mu^{-1}}(g, \phi_1) \overline{\theta_{\psi^{-1}, \mu^{-1}}(h, \phi_2)} \mathrm{d}g \mathrm{d}h,$$

where $\theta_{\psi^{-1}, \mu^{-1}}(g, \phi_i)$ is the theta series attached to ϕ_i $(i = 1, 2)$. When we are dealing with different unitary groups, we write $\mathscr{I}^V$ to indicate that the distribution is on the group $\mathrm{U}(V)(\mathbb{A}')$. In order to handle the product to theta series, we introduce the following Fourier transform, which is the global counterpart of the Fourier transform (4.1.1),

$$(4.2.2) \qquad (\phi_1 \otimes \phi_2)^{\ddagger}(w^\vee) = \int\limits_{\mathbf{L}(\mathbb{A}')} \phi_1(x + z)\phi_2(x - z)\psi(\langle z, y \rangle) \mathrm{d}z.$$

The pairing is the one between $\mathbf{L}$ and $\mathbf{L}^\vee$, and $w^\vee = (x, y)$ where $x \in \mathbf{L}$ and $y \in \mathbf{L}^\vee$.

We then have

Lemma 4.2.1. *Let $\omega^{\ddagger}$ be the representation of $\mathrm{U}(V)(\mathbb{A}')$ on $\mathcal{S}(V(\mathbb{A}))$ by right translation. It gives an intertwining map*

$$-^{\ddagger} : \mathcal{S}(\mathbf{L}(\mathbb{A}'))^{\otimes 2} \to \mathcal{S}(V(\mathbb{A})),$$

where the LHS is view as a representation of $\mathrm{U}(V)(\mathbb{A}')$ by $\omega_{\psi, \mu} \otimes \omega_{\psi^{-1}, \mu^{-1}}$. Moreover, we have

$$\theta_{\psi, \mu}(g, \phi_1)\theta_{\psi^{-1}, \mu^{-1}}(g, \phi_2) = \sum_{v^\vee \in V^\vee(k)} (\phi_1 \otimes \phi_2)^{\ddagger}(v^\vee g).$$

Proof. Suppose $(V, \langle -, - \rangle_V)$ is a Hermitian space. We denote by $(-V)$ the Hermitian space with the same underline vector space and the Hermitian form $-\langle -, - \rangle_V$. There is a commutative diagram of group embeddings

$$
\begin{array}{ccccc}
\mathrm{U}(V) & \xrightarrow{\Delta} & \mathrm{U}(V) \times \mathrm{U}(V) & \xrightarrow{\iota_\mu \times \mu^{-1}\iota_\mu} & \mathrm{Mp}(\mathrm{Res}\, V) \times \mathrm{Mp}(\mathrm{Res}(-V)) \\
& & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \tilde{i}} \\
& & \mathrm{U}(V + (-V)) & \xrightarrow{\iota_\mu} & \mathrm{Mp}(\mathrm{Res}\, V + \mathrm{Res}(-V)),
\end{array}
$$

where we identify $\mathrm{U}(V)$ with $\mathrm{U}(-V)$ in an obvious way. Let Ω_ψ be the Weil representation of $\mathrm{Mp}(\mathrm{Res}\, V + \mathrm{Res}(-V))$. It is shown in [HKS1996] that $\Omega_\psi \circ \tilde{i} = \omega_\psi \boxtimes \omega_{\psi^{-1}}$. Thus by the commutativity of the diagram,

$$\omega_{\psi, \mu} \otimes \omega_{\psi^{-1}, \mu^{-1}} = \Omega_\psi \circ \iota_\mu \circ i \circ \Delta.$$

On the other hand, the subspace V diagonally embedded in $V + (-V)$ gives a Lagrangian subspace of $V + (-V)$. The Weil representation $\omega_{\psi, \mu} \otimes \omega_{\psi^{-1}, \mu^{-1}}$ is also realized on $\mathcal{S}(V^\vee(\mathbb{A}))$ and is given by

$$\omega_{\psi, \mu} \otimes \omega_{\psi^{-1}, \mu^{-1}}(h)\phi(x) = \mu(\det h)\phi(xh)$$

for any function $\phi \in \mathcal{S}(V^\vee(\mathbb{A}))$.

The intertwining map between two realizations is given by the partial Fourier transform (4.2.2). The product formula of the theta series is then given by the Poisson summation formula. See also [HKS1996]. $\square$

Similar to the general linear group side, we say a test function $(f, \phi_1 \otimes \phi_2)$ is good, if the following conditions hold.

(1) There is a non-archimedean place v_1 of k' that splits in k such that f_{v_1} is a truncated matrix coefficient of a supercuspidal representation. This means
$$f = \widetilde{f} \cdot \mathbf{1}_{G(V)^*(k'_{v_1})}.$$
Here since the place v_1 splits in k, the group $G(V)(k'_{v_1}) \simeq \mathrm{GL}_n(k'_{v_1}) \times \mathrm{GL}_n(k'_{v_1})$. The function $\widetilde{f}$ is a matrix coefficient of a supercuspidal representation. We define
$$G(V)^*(k_{v_1}) = \{(g_1, g_2) \in G(V)(k'_{v_1}) \mid |\det g_1| = |\det g_2| = 1\},$$
and $\mathbf{1}_{G(V)^*(k'_{v_1})}$ stands for the characteristic function of the set $G(V)^*(k'_{v_1})$. In particular, f is compactly supported.
(2) There is another non-archimedean place $v_2 \neq v_1$ that splits in k, such that $\Psi_{f,\phi_1 \otimes \phi_2}$ is supported on the regular semisimple locus of $\mathsf{Y}(k'_{v_2})$.
(3) For any archimedean place v, the function f_v is K_v-finite where K_v is the maximal compact subgroup of $G(V)(k'_v)$.

Proposition 4.2.2. *For a good test function $(f, \phi_1 \otimes \phi_2)$, the integral (4.2.1) defining the distribution $\mathscr{J}$ is absolutely convergent. Moreover, if f, ϕ_1, ϕ_2 are all factorizable, then*
$$\mathscr{J}(f, \phi_1 \otimes \phi_2) = \sum_{[\zeta, w]} \prod_v \mathbf{O}_v^{\psi, \mu}([\zeta, w^\vee], f_v, \phi_{1,v} \otimes \phi_{2,v}),$$
where $\mathbf{O}_v^{\psi, \mu}$ is the orbital integral defined in Section 4.1. The right hand side is a sum over regular semisimple orbits in $G(V)(k') \times V^\vee$. Moreover it is absolutely convergent.

Proof. By Lemma 4.2.1, the distribution $\mathscr{J}$ can be written as
$$(4.2.1) = \iint_{(H(V)(k')\backslash H(V)(\mathbb{A}'))^2} \sum_{\zeta \in G(V)(k')} \sum_{w^\vee \in V^\vee(k')} f(g^{-1}\zeta h)\left(\omega_{\psi^{-1}, \mu^{-1}}(h^{-1}\zeta_1^{-1}g)\phi_1 \otimes \phi_2\right)^\ddagger(w^\vee h)\mathrm{d}g\mathrm{d}h.$$
Changing order of summation and integration, we get
$$(4.2.1) = \sum_{[\zeta, w^\vee]} \iint_{(H(V)(\mathbb{A}'))^2} f(g^{-1}\zeta h)\left(\omega_{\psi^{-1}, \mu^{-1}}(h^{-1}\zeta_1^{-1}g)\phi_1 \otimes \phi_2\right)^\ddagger(w^\vee h)\mathrm{d}g\mathrm{d}h,$$
where the sum is over all the regular semisimple orbits in $G(V)(k') \times V^\vee$. Using the same argument as in Proposition 3.2.3, one can show that the global orbital integral is absolutely convergent and the outer sum is absolutely convergent. $\qquad\square$

5. COMPARISON OF TWO RELATIVE TRACE FORMULAE

5.1. **Orbits.** In this section, we review some of the properties of the orbit spaces on both relative trace formulae, c.f. [Liu, § 5.3].

Let us introduce the affine space
$$\mathsf{M} = \mathrm{Mat}_{n,n} \times k_n \times k^n,$$
and the right action of $\mathrm{GL}_n(k)$, given by $[\xi, v^\vee, v].h = [h^{-1}\xi h, v^\vee h, h^{-1}v]$. The categorical quotient $\mathsf{M} /\!\!/ \mathrm{GL}_n$ is isomorphic to the affine space of dimension $2n$, and the isomorphism is given by taking invariants:
$$[\xi, v^\vee, v] \mapsto \{(\mathrm{Tr}\wedge^i\xi)_{i=1,\cdots,n},\ (v^\vee\xi^{j-1}v)_{j=1,\cdots,n}\}.$$
An element $[\xi, v^\vee, v]$ of M is regular semisimple if
$$\det(v^\vee\xi^{i+j}v)_{i,j=0}^{n-1} \neq 0,$$
or equivalently that ξ is regular semisimple as an element in $\mathrm{Mat}_{n,n}(k)$, and
$$\{\xi^i v \mid i = 0, \cdots, n-1\} \text{ generate } k^n,\ \{v^\vee\xi^i \mid i = 0, \cdots, n-1\} \text{ generate } k_n.$$
It is shown in [Liu, Lemma 5.5], [RS, Proposition 6.2 & Theorem 6.1] that two regular semisimple elements in M are in the same $\mathrm{GL}_n(k)$ orbit if and only if they have the same invariants.

Let $\mathsf{X} = \mathsf{S}_n(k') \times k'_n \times k'^n$ with the action of $\mathrm{GL}_n(k')$ as in Section 3 and $\mathsf{Y} = \mathrm{U}(V) \times V^\vee$ with the action of $\mathrm{U}(V)$ as in Section 4. By choosing a basis of V, we identify V (resp. $V^\vee$) with k^n (resp. k_n) and the

Hermitian form is given by a Hermitian matrix β (resp. ${}^{t}\beta$). In this way, $\operatorname{Res} V^{\vee}$ is a symplectic space via the pairing

$$\langle w, w' \rangle = \widetilde{\operatorname{Tr}}(jw^{\tau}\beta\, {}^{t}w').$$

The space X is embedded in M in an obvious way. There is also an embedding of the symmetric space $\mathsf{Y}^{V} \hookrightarrow \mathsf{M}$ via

$$[\zeta, w^{\vee}] \mapsto [\zeta, w^{\vee}, \beta^{-1}\, {}^{t}w^{\vee\tau}],$$

where the Hermitian form on V is given by the Hermitian matrix β. An element $[\gamma, v^{\vee}, v]$ in X (resp. $[\zeta, w^{\vee}]$ in Y) is regular semisimple if and only if it is regular semisimple in M via the embeddings above. Denote by $\mathsf{X}_{\mathrm{rss}}$ (resp. $\mathsf{Y}_{\mathrm{rss}}$) the subset of regular semisimple elements in X (resp. Y).

Lemma 5.1.1 ([Liu, Lemma 5.5 & 5.6]). *There is a bijection*

$$\mathsf{X}_{\mathrm{rss}} \,/\!\!/\, \mathrm{GL}_n(k') \simeq \coprod_{V \in \mathsf{Herm}_n(k)} \mathsf{Y}_{\mathrm{rss}}^{V} \,/\!\!/\, \mathrm{U}(V)(k').$$

Moreover two elements correspond if and only if they have the same invariants when embedded in M.

We say two regular semisimple orbits match if they correspond via the bijection in Lemma 5.1.1. We shall use the notation

$$[\xi, v^{\vee}, v] \leftrightarrow [\zeta, w^{\vee}]^{V}$$

to indicate the matching orbits.

Sometimes it is also convenient to consider the right action of $\mathsf{H}_1(k') \times \mathsf{H}_2(k')$ on $\mathsf{G}(k') \times k'_n \times k'^n$ and the right action of $H(V)(k') \times H(V)(k')$ on $G(V)(k') \times V^{\vee}$. From Lemma 5.1.1, one sees that there is a bijection between the regular semisimple orbits of

$$\mathsf{G}(k') \times k'_n \times k'^n,$$

and the regular semisimple orbits of

$$G(V)(k') \times V^{\vee},$$

when V ranges over all isomorphism classes of Hermitian spaces of dimension n. Let $[\xi, v^{\vee}, v] \in \mathsf{G}(k') \times k'_n \times k'^n$ and $[\zeta, w^{\vee}]^{V} \in G(V)(k') \times V^{\vee}$. We shall use the same notation

$$[\xi, v^{\vee}, v] \leftrightarrow [\zeta, w^{\vee}]^{V}$$

to indicate matching orbits.

5.2. **Smooth transfer.** In this section, we are in the local situation.

Following [Liu, § 5.4], one defines a transfer factor $\mathbf{t}([\xi, x, y])$ as follows. Let

$$\mathbf{T}_{[\xi,x,y]} = \det \begin{pmatrix} x \\ x\xi \\ \vdots \\ x\xi^{n-1} \end{pmatrix},$$

and define the transfer factor

$$\mathbf{t}([\xi, x, y]) = \mu\left(\mathbf{T}_{[\xi,x,y]} \cdot (\det \xi)^{-\left[\frac{n}{2}\right]}\right),$$

when $[\xi, x, y]$ is regular semisimple. Here the function val is the valuation on the local field k'. We define the transfer factor on $\mathsf{G}(k') \times k'_n \times k'^n$ by composing $\mathbf{t}$ with the natural map

$$\mathsf{G}(k') \times k'_n \times k'^n \to \mathsf{X}(k'), \quad [(\xi_1, \xi_2), x, y] \mapsto [\xi_1^{-1}\xi_2, x, y].$$

We also denote it by $\mathbf{t}$ by abuse of notation.

Let (F, Φ) be a test function on the general linear group side and $(f^{V}, \phi_1^{V} \otimes \phi_2^{V})$ a test function for each $V \in \mathsf{Herm}_n(k)$ on the unitary group side. We say that (F, Φ) and $\{(f^{V}, \phi_1^{V} \otimes \phi_2^{V})\}_{V \in \mathsf{Herm}_n(k)}$ are smooth transfer of each other if for all matching orbits

$$[\xi, v^{\vee}, v] \leftrightarrow [\zeta, w^{\vee}]^{V},$$

one has

$$\mathbf{t}([\xi, v^{\vee}, v])\mathbf{O}^{\psi,\mu}([\xi, v^{\vee}, v], F, \Phi) = \mathbf{O}^{\psi,\mu,V}([\zeta, w^{\vee}]^{V}, f^{V}, \phi_1^{V} \otimes \phi_2^{V}).$$

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The goal of this section is to prove the following

Proposition 5.2.1. *Let k' be non-archimdean. The smooth transfer exists. That is, for any test function (F, Φ), there exists a test function $(f^V, \phi_1^V \otimes \phi_2^V)$ for each $V \in \mathsf{Herm}_n(k)$, such that $\{(f^V, \phi_1^V \otimes \phi_2^V)\}_{V \in \mathsf{Herm}_n(k)}$ is the smooth transfer of (F, Φ). The other direction also holds.*

The case of k' being archimedean and $k = k'_\circ \times k'_\bullet$ will be treated in the next section by explicit computations.

From Lemma 3.1.2 and 4.1.2, in order to prove Proposition 5.2.1, we only need to prove the following simplified version.

Let $\Upsilon \in \mathcal{S}(\mathsf{X})$ be a test function and $\Psi^V \in \mathcal{S}(\mathsf{Y}^V)$ a collection of test functions on $\{\mathsf{Y}^V\}_{V \in \mathsf{Herm}_n(k)}$. We say they are smooth transfer of each other if for any matching orbits $\mathsf{x} \in \mathsf{X}(k')$ and $\mathsf{y}^V \in \mathsf{Y}^V(k')$, we have

$$\mathbf{t}(\mathsf{x})\mathbf{O}(\mathsf{x}, \Upsilon) = \mathbf{O}^V(\mathsf{y}^V, \Psi^V).$$

Here the left (resp. right) hand side is the simplified orbital integral on the general linear group (resp. unitary group) defined by equation (3.1.3) (resp. (4.1.4)).

Proposition 5.2.2. *The smooth transfer for the simplified orbital integral exists. That is, for any test function $\Upsilon \in \mathcal{S}(\mathsf{X})$, there exist a collection of test functions $\{\Psi^V \in \mathcal{S}(\mathsf{Y}^V)\}_{V \in \mathsf{Herm}_n(k)}$ which is the smooth transfer of Υ. The other direction also holds.*

We shall deduce Proposition 5.2.2 from the existence of smooth transfer on the level of Lie algebra which we now introduce.

Let $\mathfrak{s}_n$ (resp. $\mathfrak{u}(V)$) be the Lie algebra of S_n (resp. the unitary group $\mathrm{U}(V)$), i.e. the set of matrices

$$\{x \in \mathfrak{gl}_n(k) \mid x + x^\tau = 0\},$$

resp. the set of matrices

$$\{x \in \mathfrak{gl}_n(k) \mid x + \beta^{-1}\, {}^t x^\tau \beta = 0\},$$

where β is the Hermitian matrix defining the Hermitian structure on V. Denote $\mathfrak{x} = \mathfrak{s}_n \times k'_n \times k'^n$ and $\mathfrak{y}^V = \mathfrak{u}(V) \times V^\vee$. The group GL_n (resp. $\mathrm{U}(V)$) acts on $\mathfrak{x}$ (resp. $\mathfrak{y}^V$). For any element $[x, v^\vee, v] \in \mathfrak{x}$, $[y, w^\vee]^V \in \mathfrak{y}^V$ and any $h \in \mathrm{GL}_n(k')$, the action is defined by

$$[x, v^\vee, v].h = [h^{-1}xh, v^\vee h, h^{-1}v], \quad [y, w^\vee].h = [h^{-1}yh, w^\vee h].$$

The symmetric spaces $\mathfrak{x}$ and $\mathfrak{y}$ embed naturally in M as in the case of groups. One can also define the regular semisimple elements. An element in $\mathfrak{x}$ (resp. $\mathfrak{y}$) is regular semisimple if its image in M is regular semisimple. The regular semisimple elements form a Zariski dense open subset of $\mathfrak{x}$ (resp. $\mathfrak{y}^V$), and is denoted by $\mathfrak{x}_{\mathrm{rss}}$ (resp. $\mathfrak{y}^V_{\mathrm{rss}}$). The following lemma is [Zha, Lemma 3.1].

Lemma 5.2.3. *There is a bijection*

$$\mathfrak{x}_{\mathrm{rss}} /\!\!/ \mathrm{GL}_n(k') \simeq \coprod_{V \in \mathsf{Herm}_n(k)} \mathfrak{y}^V_{\mathrm{rss}} /\!\!/ \mathrm{U}(V)(k').$$

Two elements correspond if and only if they have the same invariants when embedded in M. More intrinsically, the categorical quotients $\mathfrak{x} /\!\!/ \mathrm{GL}_n$ and $\mathfrak{y}^V /\!\!/ \mathrm{U}(V)$ are isomorphic.

The transfer factor on the level of Lie algebra is defined as follows. Let $[\xi, x, y] \in \mathfrak{x}_{\mathrm{rss}}(k')$ be a regular semisimple element. Let

$$\mathfrak{T}_{[\xi,x,y]} = \det \begin{pmatrix} x \\ x\xi/\tau \\ \vdots \\ x(\xi/\tau)^{n-1} \end{pmatrix},$$

and define the transfer factor

$$\mathbf{t}([\xi, x, y]) = \eta\left(\mathfrak{T}_{[\xi,x,y]}\right).$$

Let $\Upsilon \in \mathcal{S}(\mathfrak{x})$ be a test function on $\mathfrak{x}$ and $\{\Psi^V \in \mathcal{S}(\mathfrak{y}^V)\}_{V \in \mathsf{Herm}_n(k)}$ be a collection of test functions on $\{\mathfrak{y}^V\}_{V \in \mathsf{Herm}_n(k)}$. For $x \in \mathfrak{x}$ (resp. $y^V \in \mathfrak{y}^V$), one can define the orbital integral $\mathbf{O}(x, \Upsilon)$ (resp. $\mathbf{O}^V(y^V, \Psi^V)$)

in the same way as the orbital integral on groups. We say that two test function Υ and Ψ^V are smooth transfer of each other if we have

$$\mathfrak{t}(x)\mathbf{O}(x, \Upsilon) = \mathbf{O}(y^V, \Psi^V)$$

for any matching orbits $x \in \mathfrak{x}$ and $y^V \in \mathfrak{y}^V$.

The following is a deep theorem proved by W. Zhang. This is one of the main ingredients in the proof of GGP conjecture for $\mathrm{U}(n+1) \times \mathrm{U}(n)$.

Theorem 5.2.4 ([Zha, Theorem 2.6]). *The transfer on the level of Lie algebra exists, i.e. for any test function $\Upsilon \in \mathcal{S}(\mathfrak{x})$, there exist a collection of test functions $\Psi^V \in \mathcal{S}(\mathfrak{y}^V)$ which is the smooth transfer of Υ. The other direction also holds.*

To prove Proposition 5.2.2, we introduce yet another transfer problem. Let $\nu \in k$ and define

$$D_\nu = \{x \in \mathrm{Mat}_{n,n} \mid \det(x - \nu) \neq 0\}.$$

In other word, D_ν is the set of matrix such that ν is not an eigenvalue. By [Zha, Lemma 3.4], for any $\epsilon \in k^{\times, 1}$, the Cayley transform

$$(5.2.1) \qquad \alpha_\epsilon : \mathrm{Mat}_{n,n}(k) \backslash D_1 \to \mathrm{GL}_n(k) \backslash D_\epsilon, \quad x \to \epsilon(1 + x)(1 - x)^{-1}$$

defines a $\mathrm{GL}_n(k')$ equivariant isomorphism between $\mathfrak{s}_n(k') \backslash D_1$ and $\mathsf{S}_n(k') \backslash D_\epsilon$. Therefore there are $\epsilon_1, \cdots, \epsilon_{n+1} \in k^{\times, 1}$ such that the image of $\mathrm{Mat}_{n,n}(k) \backslash D_1$ under the morphism $\alpha_{\epsilon_i} : i = 1, \cdots n + 1$ form an open cover of $\mathsf{S}_n(k')$. Similar statements hold for the unitary groups. Moreover, the transfer factor is compatible with the Cayley transform [Zha, Lemma 3.5].

We now introduce the following local transfer problem. For each ν, let $\Upsilon \in \mathcal{S}((\mathfrak{s}_n(k') \backslash D_\nu) \times k'_n \times k'^n)$ and $(\Psi^V \in \mathcal{S}((\mathfrak{u}(V)(k') \backslash D_\nu) \times V^\vee))$ be the test functions. We say they are transfer of each other if they are transfer of each other on the level of Lie algebras.

Proposition 5.2.5. *The local transfer exists. More precisely, for any $\nu \in k$ and any test function $\Upsilon \in \mathcal{S}((\mathfrak{s}_n(k') \backslash D_\nu) \times k'_n \times k'^n)$, we can find a collection of test functions $\{\Psi^V \in \mathcal{S}((\mathfrak{u}(V)(k') \backslash D_\nu) \times V^\vee)\}_{V \in \mathsf{Herm}_n(k)}$ which is the transfer of Υ. The other direction also holds.*

Proof. We shall deduce the proposition from Theorem 5.2.4. Take a function $\Upsilon \in \mathcal{S}((\mathfrak{s}_n(k') \backslash D_\nu) \times k'_n \times k'^n)$, we shall show the existence of the local transfer. The other direction of local transfer can be proven in the same way.

Let $\widetilde{\Psi^V}$ be the transfer of Υ on the level of Lie algebra. What we want to prove is that we can take $\widetilde{\Psi^V}$ to be of compact support in $(\mathfrak{u}(V)(k') \backslash D_\nu) \times V^\vee$. As is explained in the proof of [Zha, Lemma 3.6], there is a function α_V on $(\mathfrak{u}(V)(k') \backslash D_\nu) \times V^\vee)$ with the property that

- α_V is $\mathrm{U}(V)(k')$ invariant;
- $\alpha_V = 0$ on D_ν;
- $\alpha_V(\mathsf{y}^V) = 1$ whenever the orbital integral of $\widetilde{\Psi^V}$ at y^V is not zero.

Let $\Psi^V = \alpha_V \widetilde{\Psi^V}$. Then the collection of test functions $\{\Psi^V\}_{V \in \mathsf{Herm}_n(k)}$ is the desired local transfer. $\qquad\square$

Proof of Proposition 5.2.1 and 5.2.2. It follows from the existence of Partition of Unity that Proposition 5.2.5 implies Proposition 5.2.2, hence Proposition 5.2.1. We have thus established the existence of smooth transfer. $\qquad\square$

5.3. **Smooth transfer at the split places.** At the split places (archimedean or non-archimedean), the smooth transfer can be made explicit [Liu, Proposition 5.11]. We in fact need it in a slightly different form. We are always in the local situation and assume that $k = k'_\circ \times k'_\bullet$.

Consider the test function (F, Φ) on the general linear group side. Suppose $F = F_1 \otimes F_2$. Then by [Liu, (5.14)]

$$(3.1.1) = \int_{\mathrm{GL}_n(k')} \iint_{(\mathrm{GL}_n(k'))^2} \int_{\mathrm{GL}_n(k')} \int_{k'_n} F_1(g_\circ^{-1}, g_\bullet^{-1}) F_2(h, g_\circ^{-1} g^{-1} \gamma g g_\bullet h)$$

$(5.3.1)$

$$\Phi((v^\vee g + z)g_\circ, (v^\vee g - z)g_\bullet)\mu(\det g_\circ^{-1} g_\bullet)|\det g_\circ g_\bullet|^{\frac{1}{2}} \psi'(jzg^{-1}v)\mathrm{d}z\mathrm{d}h\mathrm{d}g_\circ\mathrm{d}g_\bullet\mathrm{d}g.$$

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We make the following change of variables. First $z \mapsto zg_\circ^{-1}$, $g \mapsto gg_\circ^{-1}$, then $g_\circ \mapsto g_\bullet g_\circ$. We get

$$
\text{(5.3.2)} \qquad \text{(3.1.1)} = \int\limits_{\mathrm{GL}_n(k')} \iint\limits_{(\mathrm{GL}_n(k'))^2} \int\limits_{\mathrm{GL}_n(k')} \int\limits_{k'_n} F_1(g_\circ^{-1}g_\bullet^{-1}, g_\bullet^{-1}) F_2(h, g^{-1}\gamma g g_\circ^{-1} h)
$$

$$
\Phi(v^\vee g + z, (v^\vee g - z)g_\circ^{-1})\mu(\det g_\circ^{-1})|\det g_\circ|^{-\frac{1}{2}}\psi'(jzg^{-1}v)\mathrm{d}z\mathrm{d}h\mathrm{d}g_\circ\mathrm{d}g_\bullet\mathrm{d}g.
$$

The unitary group $\mathrm{U}(V)(k')$ is identified with $\mathrm{GL}_n(k'_\circ)$ and the Lagrangian is isomorphic to $k'_{\circ,n}$. Consider the test function $(f, \phi_1 \otimes \phi_2)$ where $f = f_1 \otimes f_2$. Then by [Liu, Proposition 5.11],

$$
\text{(4.1.2)} = \iint\limits_{(\mathrm{GL}_n(k'))^2} \int\limits_{k'_{\circ,n}} f_1(g_\circ^{-1}) f_2(g_\circ^{-1}g^{-1}\zeta g)\mu^{-1}(\det g_\circ)|\det g_\circ|^{\frac{1}{2}}\phi_1((xg+z)g_\circ)\phi_2(xg-z)\psi'(jzg^{-1}y)\mathrm{d}z\mathrm{d}g_\circ\mathrm{d}g.
$$

We make the following change of variables. Let $g \mapsto gg_\circ^{-1}$, $z \mapsto zg_\bullet^{-1}$. Then

$$
\text{(5.3.3)} \qquad \text{(4.1.2)} = \iint\limits_{(\mathrm{GL}_n(k'))^2} \int\limits_{k'_{\circ,n}} f_1(g_\circ^{-1}) f_2(g^{-1}\zeta g g_\circ^{-1})
$$

$$
\mu^{-1}(\det g_\circ)|\det g_\circ|^{-\frac{1}{2}}\phi_1(xg+z)\phi_2((xg-z)g_\circ^{-1})\psi'(jzg^{-1}y)\mathrm{d}z\mathrm{d}g_\circ\mathrm{d}g.
$$

Proposition 5.3.1. *The test function $(F_1 \otimes F_2, \Phi_\circ \otimes \Phi_\bullet)$ and $(f_1 \otimes f_2, \phi_1 \otimes \phi_2)$ match, where*

$$
\text{(5.3.4)} \qquad f_i(g) = \int\limits_{\mathrm{GL}_n(k')} F_i(gh, h)\mathrm{d}h, \quad \phi_1 = \Phi_\circ, \quad \phi_2 = \Phi_\bullet, \quad i = 1, 2.
$$

From now on, unless otherwise stated, when we speak of the smooth transfer at the split places, we always mean the one described in Proposition 5.3.1.

Corollary 5.3.2. *Suppose k' is non-archimedean. Suppose (F, Φ) and (f, ϕ) match. Then*

(1) *Suppose π is a supercuspidal representation of $\mathrm{GL}_n(k'_\circ) \times \mathrm{GL}_n(k'_\circ)$ and Π is a local base change of π. Suppose F is a truncated matrix coefficients, then so is f. Conversely, if f is a truncated matrix coefficients, one can choose F to be a truncated matrix coefficient of Π.*

(2) *The function $\Upsilon_{F,\Phi}$ is supported on the regular semisimple locus in $\mathsf{X}(k')$ if and only if $\Psi_{f,\phi}$ is supported in $\mathsf{Y}(k')$.*

Remark 5.3.3. Proposition 5.3.1 does not prove the existence of smooth transfer at the split archimedean places in general. However, the finite linear combinations of the functions of the form $\Phi_\circ \otimes \Phi_\bullet$ span a dense subset of $\mathcal{S}(V^\vee(k'))$. This is sufficient for our purposes.

5.4. **Fundamental lemma.** At the unramified places, the unramified data should match. In this section, we are always in the unramified local situation as defined in Section 1.2. We assume $k \neq k' \times k'$.

There are two elements in $\mathsf{Herm}_n(k)$, and they are distinguished by their discriminants. The one containing the self-dual lattice L is denoted by V^+, and the other is denoted by V^-. The unitary group $\mathrm{U}(V^+)$ (resp. $\mathrm{U}(V^-)$) is thus denoted by $\mathrm{U}(n)^+$(resp. $\mathrm{U}(n)^-$). The group $\mathrm{U}(n)^+$ is unramified and can be defined over $\mathfrak{o}'$ by the lattice L. Its $\mathfrak{o}'$-point $\mathrm{U}(n)^+(\mathfrak{o}')$ is a hyperspecial maximal compact subgroup.

The symmetric space S_n is also defined over $\mathfrak{o}'$. The set of $\mathfrak{o}'$-points $\mathsf{S}_n(\mathfrak{o}')$ is an open compact subset of $\mathsf{S}_n(k')$.

The fundamental lemma states that in the unramified situation above, the test function $\mathbf{1}_{\mathsf{S}_n(\mathfrak{o}')} \otimes \mathbf{1}_{\mathfrak{o}'_n} \otimes \mathbf{1}_{\mathfrak{o}'^n}$ should match $(\mathbf{1}_{\mathrm{U}_n^+(\mathfrak{o}')} \otimes \mathbf{1}_{\mathfrak{o}_n}, 0)$. Precisely, it is

Theorem 5.4.1 ([Liu]). *There is a constant $c(n)$ depending only on n, such that when the residue characteristic of k' is bigger than $c(n)$, one has*

$$\mathbf{t}([\xi, v^\vee, v]) \int\limits_{\mathrm{GL}_n(k')} \mathbf{1}_{\mathsf{S}_n(\mathfrak{o}')}(h^{-1}\xi h)\, \mathbf{1}_{\mathfrak{o}'_n}(v^\vee h)\, \mathbf{1}_{\mathfrak{o}'^n}(h^{-1}v)\eta(\det h)\mathrm{d}h$$

$$(5.4.1) \qquad = \begin{cases} \displaystyle\int\limits_{\mathrm{U}_n^+(k')} \mathbf{1}_{\mathrm{U}_n^+(\mathfrak{o}')}(h^{-1}\zeta h)\, \mathbf{1}_{\mathfrak{o}_n}(wh)\mathrm{d}h, & if\ \ [\xi, v^\vee, v] \leftrightarrow [\zeta, w],\ \zeta \in \mathrm{U}(n)^+(k') \\[2ex] 0, & if\ \ [\xi, v^\vee, v] \leftrightarrow [\zeta, w],\ \zeta \in \mathrm{U}(n)^-(k'). \end{cases}$$

For later use, we introduce the following terminology. In the unramified situation, the following test functions will be referred to as unramified test functions

$$(F, \Phi) = (\mathbf{1}_{\mathsf{G}(\mathfrak{o}')}, \mathbf{1}_{\mathfrak{o}_n}), \quad (f^{V^+}, \phi^{V^+}) = (\mathbf{1}_{G(V^+)(\mathfrak{o}')}, \mathbf{1}_{\mathbf{L}^+(\mathfrak{o}')} \otimes \mathbf{1}_{\mathbf{L}^+(\mathfrak{o}')}), \quad (f^{V^-}, \phi^{V^-}) = (0, 0).$$

Note that $\Upsilon_{F,\Phi} = \mathbf{1}_{\mathsf{S}_n(\mathfrak{o}')} \otimes \mathbf{1}_{\mathfrak{o}'_n} \otimes \mathbf{1}_{\mathfrak{o}'^n}$, $\Psi_{f^{V^+}, \phi^{V^+}} = \mathbf{1}_{\mathrm{U}_n^+(\mathfrak{o}')} \otimes \mathbf{1}_{\mathfrak{o}_n}$. The fundamental lemma claims the unramified test functions match in the unramified situation when the characteristic of the residue field is sufficiently large.

5.5. **Simple relative trace formula: spectral side.** Let us summarize the situation of the problem. By Proposition 3.2.3 and 4.2.2, for any good matching test functions

$$(F, \Phi) \leftrightarrow \{(f^V, \phi^V)\}_{V \in \mathsf{Herm}_n(k)},$$

we have an identity of the distributions

$$\mathscr{I}(F, \Phi) = \sum_{V \in \mathsf{Herm}_n(k)} \mathscr{J}^V(f^V, \phi^V).$$

We shall use in this section the following notation. Let G be a group and f be a function on G. For any element $h \in G$, we define $f^h(g) = f(gh)$.

We have the following observations.

Lemma 5.5.1. *Let v be a place of k'. Let (F_v, Φ_v) and $\{(f^V, \phi^V)\}_{V \in \mathsf{Herm}_n(k_{v'})}$ be matching test functions. Then*

$$(F^z, \Phi) \leftrightarrow \{((f^V)^{\sigma(z)}, \phi^V)\}_{V \in \mathsf{Herm}_n(k)},$$

for any $z \in Z_{\mathsf{G}(k'_v)}$, if either one of the following conditions holds

(1) *The place v splits. The matching test function (F_v, Φ_v) and $\{(f^V, \phi^V)\}_{V \in \mathsf{Herm}_n(k_{v'})}$ are related as described in Proposition 5.3.1.*

(2) *We are in the unramified situation as described in Section 5.4. The test functions are unramified.*

Proof. The case that v splits follows from a simple change of variables.

In the unramified situation, we compute (F^z, Φ) and $((f^V)^{\sigma(z)}, \phi^V)$ explicitly. To simplify notation, we suppress all the subscript v. First, on the general linear group side, with the choice of the specific test function, the integral transform (3.1.2) takes the form

$$(5.5.1) \qquad \int\limits_{\mathrm{GL}_n(k')} \int\limits_{\mathrm{GL}_n(k)} \mathbf{1}_{\mathrm{GL}_n(\mathfrak{o})}(z_1 g^{-1})\, \mathbf{1}_{\mathrm{GL}_n(\mathfrak{o})}(z_2 g^{-1}\gamma h_2)(\omega(g)\, \mathbf{1}_{\mathfrak{o}_n})^\dagger(v^{\vee,+}, v^+)\mathrm{d}g\mathrm{d}h_2.$$

By making change of variables $g \mapsto gz_1$, we get

$$(5.5.1) = \mathbf{1}_{\mathsf{S}_n(\mathfrak{o}')}(\sigma(z_1^{-1}z_2)\gamma)(\omega(z_1)\, \mathbf{1}_{\mathfrak{o}_n})^\dagger(v^{\vee,+}, v^+).$$

Since $\sigma(z_1^{-1}z_2) \in k^{\times,1} = \mathfrak{o}^{\times,1}$, we have

$$\mathbf{1}_{\mathsf{S}_n(\mathfrak{o}')}(\sigma(z_1^{-1}z_2)\gamma) = \mathbf{1}_{\mathsf{S}_n(\mathfrak{o}')}(\gamma).$$

Let us take a uniformizer ϖ of k' (hence of k). Then we can write $z_1 = \varpi'^r u_1$ where $u_1 \in \mathfrak{o}^\times$. Thus

$$(\omega(z_1)\, \mathbf{1}_{\mathfrak{o}_n})^\dagger(v^{\vee,+}, v^+) = (\omega(\varpi'^r)\, \mathbf{1}_{\mathfrak{o}_n})^\dagger(v^{\vee,+}, v^+)$$
$$= \mathbf{1}_{\mathfrak{o}'_n}(v^{\vee,+}\varpi'^r)\, \mathbf{1}_{\mathfrak{o}'^n}(\varpi'^{-r}v^+).$$

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Here we have used the fact that $\varpi' \in k'$ and u_1 fixes $\mathbf{1_o}$. Therefore the orbital integral on the general linear group side is

$$\int_{\mathrm{GL}_n(k')} \mathbf{1}_{\mathsf{S}_n(\mathfrak{o}')}(h_1^{-1}\gamma h_1)\,\mathbf{1}_{\mathfrak{o}'_n}(v^{\vee,+}\varpi'^r h_1^{-1})\,\mathbf{1}_{\mathfrak{o}'^n}(h_1\varpi'^{-r}v^+)\mathrm{d}h_1.$$

However, the invariants of the orbits $[\gamma, v^{\vee,+}, v^+]$ and $[\gamma, \varpi'^r v^{\vee,+}, \varpi'^{-r}v^+]$ are the same. This shows that

$$\mathbf{O}([\gamma, v^\vee, v], F^z, \Phi) = \mathbf{O}([\gamma, v^\vee, v], F, \Phi)$$

if we are in the unramified situation and the test functions are unramified.

We now compute the orbital integral on the unitary groups. We have $k^{\times,1} = \mathfrak{o}^{\times,1}$ and $\sigma(z) \in k^{\times,1}$. Moreover, the test function on the group is $\mathbf{1}_{\mathrm{U}(n)^+(\mathfrak{o}')} \times \mathbf{1}_{\mathrm{U}(n)^+(\mathfrak{o}')}$ (or the zero function) which is invariant under the action of elements in $k^{\times,1}$, we see that

$$\mathbf{O}([\zeta, w^\vee], (f^{V^+})^{\sigma(z)}, \phi^{V^+}) = \mathbf{O}([\zeta, w^\vee], f^{V^+}, \phi^{V^+})$$

if we are in the unramified situation the test functions are all unramified.

This proves the lemma in the case when v is unramified. $\qquad\square$

For a given pair of good matching test functions as above, let S be the finite set of places that contains $\{v_1, v_2\}$ and all the places that does not satisfy either conditions in the above lemma. Note that all the places in S except v_1, v_2 are non-split and in particular, S does not contain any archimedean places. For each $v \in S$, there is an open compact subgroup U_v of $k_v^{\times,1}$ such that if $\sigma(z) \in U_v$, then $f_v^z = f_v$.

Let $\mathbb{A}(S,1)$ be the open and closed subgroup of $\mathbb{A}^{\times,1}$ defined by

$$\{(x_v) \in \mathbb{A}^{\times,1} \mid x_v \in U_v \text{ if } x_v \in S\}.$$

Let $k(S,1)$ be the intersection of $\mathbb{A}(S,1)$ and $k^\times$. This is a discrete subgroup of $\mathbb{A}(S,1)$. Moreover, since $k^{\times,1}\backslash\mathbb{A}^{\times,1}$ is compact, the quotient group $k(S,1)\backslash\mathbb{A}(S,1)$ is compact also. Let $\mathbb{A}(S)$ (resp. $k(S)$) be the inverse image of $\mathbb{A}(S,1)$ in $\mathbb{A}^\times$ (resp. $k^\times$) under the map σ. The quotient $k(S)\mathbb{A}'^\times\backslash\mathbb{A}(S)$ is identified with $k(S,1)\backslash\mathbb{A}(S,1)$.

For any good test function (F, Φ), its translation (F^z, Φ) by element in $z \in \mathbb{A}(S)$ is also good. Similar statement holds for (f^V, ϕ^V). If

$$(F, \Phi) \leftrightarrow \{(f^V, \phi^V)\}_{V \in \mathsf{Herm}_n(k)},$$

then for any $z \in \mathbb{A}(S)$,

$$(F^z, \Phi) \leftrightarrow \{((f^V)^{\sigma(z)}, \phi^V)\}.$$

Let χ be any character of $k^{\times,1}\backslash\mathbb{A}^{\times,1}$. We conclude that

$$\int_{k(S)\mathbb{A}'^\times\backslash\mathbb{A}(S)} \mathscr{I}(F, \Phi)\chi(zz^{-\tau})\mathrm{d}z = \sum_V \int_{k(S,1)\backslash\mathbb{A}(S,1)} \mathscr{J}^V((f^V)^{\sigma(z)}, \phi^V)\chi(z)\mathrm{d}z,$$

where the sum is over $\mathsf{Herm}_n(k)$ and the measure on both sides are compatible with the isomorphism $k(S)\mathbb{A}'^\times\backslash\mathbb{A}(S) \simeq k(S,1)\backslash\mathbb{A}(S,1)$.

We now come to the spectral side of the trace formula. We treat the relative trace formula on the unitary group first.

Fix a Hermitian space V. To simplify notation, we drop all the superscript V in the expression. Let π be an irreducible cuspidal automorphic representation of $G(V)(\mathbb{A}')$. Let $(f, \phi_1 \otimes \phi_2)$ be a test function. Define the distribution

(5.5.2)
$$\mathscr{J}_\pi(f, \phi_1 \otimes \phi_2) = \sum_{\varphi \in \pi} \frac{\int_{\mathrm{U}(V)(k')\backslash\mathrm{U}(V)(\mathbb{A}')} \pi(f).\varphi(g)\theta_{\psi^{-1},\mu^{-1}}(g, \phi_1)\mathrm{d}g}{\int_{\mathrm{U}(V)(k')\backslash\mathrm{U}(V)(\mathbb{A}')} \varphi(h)\theta_{\psi^{-1},\mu^{-1}}(h, \phi_2)\mathrm{d}h}.$$

Such distribution extends by linearity to all test functions in $\mathcal{C}_c^\infty(G(V)(\mathbb{A}')) \otimes \mathcal{S}(\mathbf{L}(\mathbb{A}')) \otimes \mathcal{S}(\mathbf{L}(\mathbb{A}'))$.

Proposition 5.5.2. *If the test function $(f, \phi_1 \otimes \phi_2)$ is good, then we have the spectral expansion*

$$\int_{k(S,1)\backslash \mathbb{A}(S,1)} \mathscr{J}(f^z, \phi_1 \otimes \phi_2)\chi(z)\mathrm{d}z = \sum_\pi \mathscr{J}_\pi(f, \phi_1 \otimes \phi_2),$$

where the sum runs over all the irreducible cuspidal automorphic representations with the central character χ. Moreover, the right hand side is absolutely convergent.

To prove this proposition, we need a lemma.

Lemma 5.5.3 ([Ber, Proposition 25])**.** *Let k be an non-archimedean local field and π an irreducible admissible representation of $\mathrm{GL}_n(k)$. Let $\mathrm{GL}_n(k)^*$ be the subgroup consisting of matrices whose determinants are in $\mathfrak{o}^\times$. Then the representations $\pi\,|_{\mathrm{GL}_n(k)^*}$ is semisimple of finite length. Moreover, if π' is another irreducible admissible representation of $\mathrm{GL}_n(k)$, then the following are equivalent.*

(1) *There is an unramified character χ such that $\pi = \pi' \otimes \chi$.*
(2) *The intersection of Jordan–Holder series of $\pi|_{\mathrm{GL}_n(k)^*}$ and $\pi'|_{\mathrm{GL}_n(k)^*}$ is not empty.*

Lemma 5.5.4. *Let k be an non-archimedean local field. Let π and π' be two irreducible admissible representations of $\mathrm{GL}_n(k)$. Let $f(g)$ be a truncated matrix coefficient, i.e. $f(g) = \langle \pi(g)u, \widetilde{u}\rangle \cdot \mathbf{1}_{\mathrm{GL}_n(k)^*}$ where $\langle \pi(g)u, \widetilde{u}\rangle$ is a matrix coefficient of π. If the morphism*

$$\pi'(f) : \pi' \to \pi'$$

is not identically zero, then $\pi \simeq \pi' \otimes \chi$ for some unramified character χ. In particular, if π is supercuspidal, then so is π'.

Proof. Suppose $\pi\,\big|_{\mathrm{GL}_n(k)^*} = \oplus\pi_j$ and $\widetilde{\pi}\,\big|_{\mathrm{GL}_n(k)^*} = \oplus\widetilde{\pi}_j$. Then

$$f(g) = \sum_{i,j} f_{ij}(g) = \sum_{i,j}\langle\pi_i(g)u_i, \widetilde{u}_j\rangle,$$

with $u_i \in \pi_i$ and $\widetilde{u}_j \in \widetilde{\pi}_j$. Note that $f_{ij} = 0$ unless π_i and π_j are isomorphic. Now if $\pi'(f).v \neq 0$, for at least one f_{ij}, we have $\pi'(f_{ij}).v \neq 0$. Let $\pi'\,\big|_{\mathrm{GL}_n(k)^*} = \oplus\pi'_l$, and $v = \sum v_l$. Then one sees that for at least one l, $\pi'(f_{ij}).v_l \neq 0$. Since π_i, π_j and π'_l are all irreducible, they have to be isomorphic. In particular, the intersection of Jordan–Holder series of $\pi\,\big|_{\mathrm{GL}_n(k)^*}$ and $\pi'\,\big|_{\mathrm{GL}_n(k)^*}$ are not empty. Then by Lemma 5.5.3, we see that π and π' differ by an unramified character. $\square$

Proof of Proposition 5.5.2. The argument is similar to the simple trace formula of Arthur–Selberg. We have

$$\int_{k(S,1)\backslash\mathbb{A}(S,1)} \mathscr{J}(f, \phi_1 \otimes \phi_2)\chi(z)\mathrm{d}z$$

$$= \int_{k(S,1)\backslash\mathbb{A}(S,1)} \iint_{(\mathrm{U}(V)(k')\backslash \mathrm{U}(V)(\mathbb{A}'))^2} K(g, zh)\theta_{\psi^{-1},\mu^{-1}}(g,\phi_1)\overline{\theta_{\psi^{-1},\mu^{-1}}(h,\phi_2)}\chi(z)\mathrm{d}g\mathrm{d}h\mathrm{d}z.$$

Interchange the order of integration. The right hand side equals

$$\iint_{(\mathrm{U}(V)(k')\backslash \mathrm{U}(V)(\mathbb{A}'))^2}\left(\int_{k(S,1)\backslash\mathbb{A}(S,1)} K(g,zh)\chi(z)\mathrm{d}z\right)\theta_{\psi^{-1},\mu^{-1}}(g,\phi_1)\overline{\theta_{\psi^{-1},\mu^{-1}}(h,\phi_2)}\mathrm{d}g\mathrm{d}h.$$

The representation π that appears in the spectral expansion of

$$\int_{k(S,1)\backslash\mathbb{A}(S,1)} K(g, zh)\chi(z)\mathrm{d}z$$

satisfies the following conditions.

(1) The local component π_{v_1} at the place v_1 must be supercuspidal. The representation π is thus cuspidal. This follows from Lemma 5.5.4 and the fact that f is a truncated matrix coefficient of a supercuspidal representation.

(2) The central character ω_π of π equals χ. This is because after integrating over the center, the representation that appears in the spectral expansion satisfies the conditions that

$$\int_{k(S,1)\backslash \mathbb{A}(S,1)} \overline{\omega_\pi(z)}\chi(z)\mathrm{d}z \neq 0.$$

One then concludes that $\overline{\omega_\pi}\chi = 1$ when restricted to $\mathbb{A}(S,1)$. Since ω_π and χ are both automorphic, we conclude that $\omega_\pi = \chi$.

The proposition then follows. $\qquad\square$

Similar spectral expansion holds for the distribution on the general linear group. Let Π be an automorphic representation of $\mathsf{G}(\mathbb{A}')$ whose central character is trivial when restricted to $\mathsf{H}_2(\mathbb{A}')$. For $\varphi_\Pi \in \Pi$, if n is odd, define the following period integral

$$\mathcal{P}(\varphi_\Pi) = \int_{\mathsf{H}_2(k')\mathsf{Z}_2(\mathbb{A}')\backslash \mathsf{H}_2(\mathbb{A}')} \varphi_\Pi(h)\mathrm{d}h.$$

If n is even, define

$$\mathcal{P}(\varphi_\Pi) = \int_{\mathsf{H}_2(k')\mathsf{Z}_2(\mathbb{A}')\backslash \mathsf{H}_2(\mathbb{A}')} \varphi_\Pi(h)\eta(h)\mathrm{d}h,$$

where $\eta(h) = \eta(\det h_1 h_2)$ if $h = (h_1, h_2)$.

For good test functions (F, Φ) on the general linear group side, define

$$\mathscr{I}_\Pi(s, F, \Phi) = \sum_{\varphi_\Pi \in \Pi} Z(s, \Pi(F)\varphi_\Pi, \Phi)\mathcal{P}(\varphi_\Pi),$$

where φ_Π runs over an orthonormal basis of Π. We write simply $\mathscr{I}_\Pi(F, \Phi) = \mathscr{I}_\Pi(1/2, F, \Phi)$.

Proposition 5.5.5. *We have the spectral expansion*

$$\int_{k(S)\mathbb{A}'^{\times,1}\backslash \mathbb{A}(S)} \mathscr{I}(F^z, \Phi)\chi\left(\frac{z}{z^\tau}\right)\mathrm{d}z = \sum_\Pi \mathscr{I}_\Pi(F, \Phi),$$

where the sum runs over all the irreducible cuspidal automorphic representations whose central character is the base change of χ to $k^\times$. Moreover, the right hand side is absolutely convergent.

The proof is identical to the case of unitary groups.

5.6. **A relative trace identity.** Let us fix a supercuspidal representation π_0 of $G(V)(k'_{v_1})$ and take Π_0 to be its local base change. Suppose (F, Φ) and $\{(f^V, \phi^V)\}_{V\in \mathsf{Herm}_n(k)}$ are good test functions and are smooth transfer of each other. Assume that (F, Φ) and $\{(f^V, \phi^V)\}_{V\in \mathsf{Herm}_n(k)}$ are all factorizable. Suppose furthermore that at the place v_1, the test function F_{v_1} is a truncated matrix coefficient of Π_0. Then $f^V_{v_1}$ is a truncated matrix coefficient of π_0. Let χ be a character of $k^{\times,1}\backslash \mathbb{A}^{\times,1}$ whose component at v_1 is the central character of π_0. By Proposition 3.2.3, 5.5.5, 4.2.2 and 5.5.2, we have an identity

(5.6.1) $$\sum_\Pi \mathscr{I}_\Pi(F, \Phi) = \sum_V \sum_\pi \mathscr{J}^V_\pi(f^V, \phi^V).$$

On the left hand side, the sum runs over all cuspidal automorphic representations Π of $\mathsf{G}(\mathbb{A}')$ such that $\Pi_{v_1} \simeq \Pi_0$ and that the central character is the base change of χ. On the right hand side, the outer sum runs over the set of isomorphism classes of Hermitian spaces over k. The inner sum runs over all irreducible cuspidal automorphic representations of $G(V)(\mathbb{A}')$ such that $\pi_{v_1} \simeq \pi_0$ and the central character of π is χ. The sums of both sides are absolutely convergent.

We can choose a sufficiently large finite set S of places containing all the archimedean places and v_1, v_2, such that if $v \notin S$, then either v is split, or v is unramified and the following conditions hold.

(1) The characters ψ and χ are unramified. The element $j \in \mathfrak{o}^\times$.
(2) The characteristic of the residue field is larger than the constant $c(n)$ in Theorem 5.4.1.

With such a choice S, We choose the matching test functions
$$(F, \Phi) \leftrightarrow \{(f^V, \phi^V)\}_{V \in \mathsf{Herm}_n(k)},$$
which are unramified at the unramifieid places $v \notin S$. Note that there are only finitely many isomorphism classes of hermitian spaces V such that $(f^V, \phi^V) \neq 0$. Note also that $\mathscr{I}_\Pi(F, \Phi) \neq 0$ unless Π is unramified outside S. Similarly for $\mathscr{J}_\pi^V(f^V, \phi^V)$.

Let
$$\Lambda_{\Pi^S} : \mathcal{H}(\mathsf{G}(\mathbb{A}'^S) /\!\!/ \mathsf{K}^S) \to \mathbb{C}$$
resp.
$$\lambda_{\pi^S} : \mathcal{H}(G(V)(\mathbb{A}'^S) /\!\!/ K(V)^S) \to \mathbb{C}$$
be the character Π^S (resp. π^S), where $\mathcal{H}(\mathsf{G}(\mathbb{A}'^S) /\!\!/ \mathsf{K}^S)$ (resp. $\mathcal{H}(G(V)(\mathbb{A}'^S) /\!\!/ K(V)^S)$) is the spherical Hecke algebra. One has
$$\mathscr{I}_\Pi(F, \Phi) = \Lambda_{\Pi^S}(F^S) \mathscr{I}_\Pi(1 \otimes F_S, \Phi),$$
and
$$\mathscr{J}_\pi^V(f^V, \phi^V) = \lambda_{\pi^S}((f^V)^S)) \mathscr{J}_\pi^V(1 \otimes f_S^V, \phi^V).$$
The identity (5.6.1) can be written as

(5.6.2) $$\sum_\Pi \Lambda_{\Pi^S}(F^S) \mathscr{I}_\Pi(1 \otimes F_S, \Phi) = \sum_V \sum_\pi \lambda_{\pi^S}((f^V)^S) \mathscr{J}_\pi^V(1 \otimes f_S^V, \phi^V).$$

We fix F_S, Φ, f_S^V and ϕ^V. By our choice, (F_v, Φ_v) (resp. (f_v^V, ϕ_v^V)) is unramified if $v \notin S$. By Proposition 5.3.1, one has $\lambda_{\pi^S}(f^S) = \Lambda_{\Pi^S}(b(F^S))$, where b is the base change map of spherical Hecke algebra. We now vary the test functions at the split places of k'. The identity 5.6.2 can be viewed as an identity of the character of the split spherical Hecke algebra
$$\bigotimes_{\substack{v \notin S \\ v \text{ split}}} \mathcal{H}(\mathsf{G}(k_v') /\!\!/ \mathsf{K}_v).$$

We need two facts.
  (1) The characters of the split spherical Hecke algebra of different representations are linearly independent.
  (2) By a theorem of Ramakrishnan [Ram, Theorem A], given for almost all split places v an irreducible admissible representation $\pi_v^\diamond$, there is at most one cuspidal automorphic representation $\Pi^\diamond$ of $\mathsf{G}(\mathbb{A}')$, such that at almost all split places v, $\Pi_v^\diamond$ is the local base change of $\pi_v^\diamond$.

We can thus write the identity (5.6.2) as

(5.6.3) $$\sum_\Pi \left(\Lambda_{\Pi^S}(F^S) \mathscr{I}_\Pi(1 \otimes F_S, \Phi) - \sum_V \sum_{\substack{\pi \\ \Pi_v \simeq \mathrm{BC}(\pi_v) \\ v \notin S \\ v \text{ split}}} \lambda_{\pi^S}((f^V)^S) \mathscr{J}_\pi^V(1 \otimes f_S^V, \phi^V) \right) = 0.$$

Now fix an irreducible cuspidal automorphic representation $\pi^\#$ of $U(V)(\mathbb{A}')$ whose local component at v_1 is π_0 and is unramified at the places outside S. Let $\Pi = \mathrm{BC}(\pi^\#)$. We then deduce from the linear independence of the characters and identity (5.6.3) that

(5.6.4) $$\mathscr{I}_\Pi(F, \Phi) = \sum_V \sum_{\substack{\pi \\ \Pi \simeq \mathrm{BC}(\pi_v) \\ v \notin S \\ v \text{ split}}} \mathscr{J}_\pi^V(f^V, \phi^V),$$

By the theorem of Ramakrishnan [Ram, Theorem A], Π is in fact the weak base change of any π appearing on the right hand side of identity (5.6.4). Thus we know that the representations π appearing on the right hand side are all nearly equivalent to $\pi^\#$, and the components at v_1 are all π_0. In summary, we have the following relative trace identity.

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Proposition 5.6.1. *Let the notation be as above. Let π be an irreducible cuspidal automorphic representation $G(V)(\mathbb{A}')$ such that at the nonarchimedean split places v_1 and v_2, the local components of π is supercuspidal. Let Π be the weak base change of π. Then for the matching good test functions (F, Φ) and $\{(f^V, \phi^V)\}_{V \in \mathsf{Herm}_n(k)}$, we have*

$$(5.6.5) \qquad \mathscr{I}_\Pi(F, \Phi) = \sum_V \sum_{\substack{\pi' \\ \pi' \text{ nearly equivalent to } \pi}} \mathscr{J}_{\pi'}^V(f^V, \phi^V).$$

6. Proof of Theorem 1.1.1

6.1. **Local distributions.** Let π be an irreducible cuspidal automorphic representation of $G(V)(\mathbb{A}')$. Recall that the local multiplicity one theorem [Sun2012, SZ2012] tells us that

$$\dim \mathrm{Hom}_{H(V)(k_v')}(\pi_v \otimes \omega_{\psi_v^{-1}, \mu_v^{-1}}, \mathbb{C}) \leq 1.$$

Fix a generator ℓ_v at each place. We adapt the convention $\ell_v = 0$ if the space is zero. Then one can decompose the global Fourier–Jacobi period of the unitary group, hence the distribution $\mathscr{J}_\pi$ as a product of the local ones when the test functions are all factorizable. More precisely, we fix for each place v an inner product $\langle -, - \rangle_v$ on π_v such that their product equals the global Peterson inner product. The global Fourier–Jacobi period decomposes as

$$\mathcal{FJ}_{\psi, \mu} = c_\pi \prod_v \ell_v,$$

where c_π is a constant depending on π and its realization in the cuspidal automorphic spectrum of $G(V)(\mathbb{A}')$. We define the local spherical character

$$\mathscr{J}_{\pi_v}(f_v, \phi_{1,v} \otimes \phi_{2,v}) = \sum_{\varphi_v} \ell_v(\pi_v(f_v)(\varphi_v), \phi_{1,v}) \overline{\ell_v(\varphi_v, \phi_{2,v})},$$

where the sum runs over a set of orthonormal basis of π_v. The distribution then decomposes as

$$\mathscr{J}_\pi(f, \phi_1 \otimes \phi_2) = |c_\pi|^2 \prod_v \mathscr{J}_{\pi_v}(f_v, \phi_{1,v} \otimes \phi_{2,v}).$$

We say a test function is of positive type if it is a finite sum of the test functions (f, ϕ) of the following form

(1) There is a function $f_0 \in \mathcal{S}(G(V)(\mathbb{A}'))$ such that

$$f = f_0 * f_0^\star,$$

where $f_0^\star(g) = \overline{f_0(g^{-1})}$.
(2) The Schwartz function $\phi = \phi_0 \otimes \phi_0$ where $\phi_0 \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$.

It is clear that for a test function (f, ϕ) of positive type, one has

$$\mathscr{J}_\pi(f, \phi) \geq 0.$$

We also have the analogous notion of positive type in the local situation.

Similar decomposition of the distribution also holds for the distribution $\mathscr{I}_\Pi$. As before we fix an inner product on Π_v for each place v. Let ω_v be the Weil representation of $\mathrm{GL}_n(k)$ as defined in Section 2.1. Since $\mathrm{Hom}_{\mathrm{GL}_n(k)}(\Pi_v \otimes \overline{\omega}_v, \mathbb{C})$ is one dimensional, we choose the local linear functionals $Z_v^* \in \mathrm{Hom}_{\mathrm{GL}_n(k)}(\Pi_v \otimes \overline{\omega}_v, \mathbb{C})$ such that the Rankin–Selberg integral decomposes as $Z = \prod_v Z_v^*$. If the representation Π_v is supercuspidal at the place v, then we take $Z_v^* = Z_v$, where Z_v is the local Rankin–Selberg integral evaluated at $s = 1/2$ [JPSS1983]. By the explicit decomposition in [GJR2001, § 2, page 184-187], the linear form $\mathcal{P}$ (c.f. § 5.5) is a product of local linear forms $\mathcal{P}_v$. It is denoted by $\mathcal{J}_v$ in [GJR2001, page 185]. Let (F_v, Φ_v) be a test function. We define the local distribution $\mathscr{I}_{\Pi_v}$ by

$$\mathscr{I}_{\Pi_v}(F_v, \Phi_v) = \sum_{\varphi_v} Z_v^*(\Pi(F_v)(\varphi_v), \Phi_v) \mathcal{P}_v(\varphi_v),$$

where the sum runs over a set of orthonormal basis of Π_v. One has thus

$$\mathscr{I}_\Pi(F, \Phi) = c'_\Pi \prod_v \mathscr{I}_{\Pi_v}(F_v, \Phi_v)$$

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for some constant c'_Π depending on the representation Π.

Lemma 6.1.1. *Let v be a split place of k'. Assume the representation Π_v is a local base change of π_v. Suppose (F_v, Φ_v) and $\{(f_v, \phi_v)\}$ are smooth transfer of each other. Then*

$$\mathscr{I}_{\Pi_v}(F_v, \Phi_v) = c\mathscr{J}_{\pi_v}(f_v, \phi_v)$$

for some nonzero constant c.

Proof. As we are looking at only one place v, we drop all the subscripts v. As Π is the base change of π, one has $\Pi = \pi \boxtimes \bar{\pi}$. Here we have identified $\bar{\pi}$ with the contragredient of π via the fixed invariant bilinear pairing. Let $\{\varphi_i\}$ be an orthonormal basis of π. The local linear functional $\mathcal{P}$ is nothing but a constant multiple of the invariant pairing between π and $\bar{\pi}$. Then

$$\mathcal{P}(\varphi_i \otimes \varphi_j) \neq 0$$

if and only if $i = j$ and this number does not depend on i, j but only on the choice of $\mathcal{P}$. One then note that at the split place, Z^* is a nonzero constant multiple of $\ell \otimes \bar{\ell}$. The lemma then follows. $\qquad\square$

Proposition 6.1.2. *Suppose v' is a split place and $\pi_{v'}$ is supercuspidal. Assume that $\mathscr{J}_{\pi_{v'}}$ is nonzero. Then there is a test function*

$$(f_{v'}, \phi_{v'}),$$

with $f_{v'} \in \mathcal{S}(\mathrm{GL}_n(k'_{v'}) \times \mathrm{GL}_n(k'_{v'}))$, $\phi_{v'} = \sum \phi_j \otimes \phi'_j$, and $\phi_j, \phi'_j \in \mathcal{S}(k'_{v',n})$, such that

(1) $\mathscr{J}_{\pi_{v'}}(f_{v'}, \phi_{v'}) \neq 0$.
(2) *The integral transform* (4.1.3) *$\Psi_{f_{v'}, \phi_{v'}}$ is supported on the regular semisimple locus.*

The proof of this proposition is rather technical. In order not to interrupt the proof of our main theorem, we postpone the proof of it to the next section.

6.2. **Proof: (2) implies (1).** Let π be an irreducible cuspidal automorphic representation of $G(V)(\mathbb{A}')$. We may assume that π has a nonzero Fourier–Jacobi period $\mathcal{FJ}_{\psi,\mu}$. Let Π be its weak base change. In order to prove (1), it is enough to show that there is a choice of test functions (F, Φ), such that

$$\mathscr{I}_\Pi(F, \Phi) \neq 0.$$

By the definition of the weak base change, all the representation π which are nearly equivalent have the same components at the places v_1 and v_2. At these two places, $G(V)_{v_i}$ is identified with $\mathrm{GL}_n(k'_{v_i}) \times \mathrm{GL}_n(k'_{v_i})$. We denote them by $G_i, i = 1, 2$ respectively to simplify notation. Moreover we denote by G_1^* the subgroup of G_1 consisting of matrices whose determinant is in $\mathfrak{o}'^\times_{v_1} \times \mathfrak{o}'^\times_{v_1}$.

Since the global Fourier–Jacobi period of π is not identically zero, one can find an automorphic form φ and a Schwartz function $\widetilde{\phi}$ such that $\mathcal{FJ}(\varphi, \widetilde{\phi}) \neq 0$. Moreover, one can take $\varphi = \otimes_v \varphi_v$ and $\widetilde{\phi} = \otimes_v \widetilde{\phi}_v$ both factorizable. This implies that the local functionals ℓ_v are all nonzero, i.e. we can find $\varphi_v \in \pi_v$ and Schwartz function $\widetilde{\phi}_v$, such that $\ell_v(\varphi_v, \phi_v) \neq 0$.

We now show that at the place v_1, one can find a test function (f_{v_1}, ϕ_{v_1}) such that the local distribution $\mathscr{J}_{\pi_{v_1}}(f_{v_1}, \phi_{v_1}) \neq 0$, and f_{v_1} is a truncated matrix coefficient of π_{v_1} as defined in Lemma 5.5.3.

Let f_{v_1} be the truncated matrix coefficient

$$f_{v_1}(g) = \langle \pi_{v_1}(g^{-1})\varphi_{v_1}, \varphi_{v_1} \rangle \mathbf{1}_{G_1^*}(g).$$

It defines a linear map

$$\pi_{v_1}(f_{v_1}) : \pi_{v_1} \to \pi_{v_1}.$$

We claim that $\pi_{v_1}(f_{v_1})$ is a self-adjoint operator. The adjoint operator of $\pi_{v_1}(f_{v_1})$ is $\pi_{v_1}(f^\star_{v_1})$, where $f^\star_{v_1}(g) = \overline{f_{v_1}(g^{-1})}$. Since

$$\langle \pi(g^{-1})\varphi_{v_1}, \varphi_{v_1} \rangle = \langle \varphi_{v_1}, \pi(g)\varphi_{v_1} \rangle = \overline{\langle \pi(g)\varphi_{v_1}, \varphi_{v_1} \rangle},$$

we conclude that $f_{v_1}(g) = f^\star_{v_1}(g)$. The claim then follows.

Since $\pi_{v_1}(f_{v_1})$ is self-adjoint, its image and kernel are orthogonal. Moreover, the function f_{v_1} is of positive type since

$$f_{v_1} = f_{v_1} * f^\star_{v_1}$$

up to a nonzero constant.

We take a basis of π as follows. Take an orthonormal basis $\{b_1, \cdots, b_r\}$ of the image of $\pi_{v_1}(f_{v_1})$ and an orthonormal basis of the kernel of $\pi_{v_1}(f_{v_1})$. They form an orthonormal basis of π_{v_1}. Then up to some positive constant,

$$\mathscr{I}_{\pi_{v_1}}(f_{v_1}, \widetilde{\phi_{v_1}} \otimes \widetilde{\phi_{v_1}}) = \sum_{j=1}^{r} \ell_{v_1}(\pi_{v_1}(f_{v_1})b_j, \widetilde{\phi_{v_1}})\overline{\ell_{v_1}(b_j, \widetilde{\phi_{v_1}})} = \sum_{j=1}^{r} |\ell_{v_1}(b_j, \widetilde{\phi_{v_1}})|^2 > 0.$$

Since the Fourier–Jacobi period of π is not identically zero, one can find an $(\widetilde{f}, \phi \otimes \phi)$ which is factorizable and is of positive type, such that

$$\mathscr{I}_{\pi}(\widetilde{f}, \widetilde{\phi} \otimes \widetilde{\phi}) > 0.$$

We want to modifiy $\widetilde{f}$ at the places v_1 and v_2 and $\widetilde{\phi}$ at v_2 to get a desired test function.

At the place v_1, let f_{v_1} be the truncated matrix coefficient

$$\langle \pi_{v_1}(g^{-1})\varphi_{v_1}, \widetilde{\varphi_{v_1}} \rangle \mathbf{1}_{G_1^*}(g),$$

with $\widetilde{\varphi_{v_1}} \in \widetilde{\pi_1}$. Then

$$\mathscr{I}_{\pi_{v_1}}(f_{v_1}, \widetilde{\phi_{v_1}} \otimes \widetilde{\phi_{v_1}}) > 0.$$

At the place v_2, Proposition 6.1.2 guarantees that we can choose a test functions (f_{v_2}, ϕ_{v_2}) such that $\Psi_{f_{v_2}, \phi_{v_2}}$ is supported on the regular semisimple locus and

$$\mathscr{I}_{\pi_{v_2}}(f_{v_2}, \phi_{v_2}) \neq 0, \quad \phi_{v_2} = \sum_{j} \phi_{1,v_2}^{(j)} \otimes \phi_{2,v_2}^{(j)}.$$

Define the new test functions

$$f = \prod_{v \neq v_1, v_2} \widetilde{f}_v \times f_{v_1} \times f_{v_2},$$

and

$$\phi = \prod_{v \neq v_2} (\widetilde{\phi}_v \otimes \widetilde{\phi}_v) \times \left(\sum_{j} \phi_{1,v_2}^{(j)} \otimes \phi_{2,v_2}^{(j)} \right).$$

We choose the test function (F, Φ) to be the smooth transfer of $\{(f^V, \phi_1^V \otimes \phi_1^V)\}_{V \in \mathsf{Herm}_n(k)}$, where $(f^{V'}, \phi_1^{V'} \otimes \phi_2^{V'}) = 0$ if $V' \not\simeq V$ and $(f^V, \phi^V) = (f, \phi)$. These test functions are good, we can thus apply the relative trace identity. With our choice of the test functions, identity (5.6.5) is reduced to

$$(6.2.1) \qquad \mathscr{I}_{\Pi}(F, \Phi) = \sum_{\pi' \text{ nearly equivalent to } \pi} \mathscr{I}_{\pi}(f, \phi_1 \otimes \phi_2).$$

The right hand side of identity (6.2.1) can be written as

$$\mathscr{I}_{\Pi}(F, \Phi) = \mathscr{I}_{\pi_{v_1}}(f_{v_1}, \phi_{v_1}) \mathscr{I}_{\pi_{v_2}}(f_{v_2}, \phi_{v_2}) \left(\sum_{\pi'} \mathscr{I}_{\pi'}^{v_1, v_2}(f^{v_1, v_2}, \widetilde{\phi}^{v_1, v_2} \otimes \widetilde{\phi}^{v_1, v_2}) \right),$$

where

$$f^{v_1, v_2} = \prod_{v \neq v_1, v_2} f_v$$

and similarly for $\widetilde{\phi}^{v_1, v_2}$ and $\mathscr{I}_{\pi'}^{v_1, v_2}$. On the right hand side of this identity, the factors outside the parentheses are nonzero. In the sum inside the parentheses, all the terms are non-negative since the test functions are all of positive type. Moreover there is at least one strictly positive term (which corresponds to π). We conclude that $\mathscr{I}_{\Pi}(F, \Phi) \neq 0$.

6.3. **Proof: (1) implies (2).** The nonvanishing of the L-function at $s = \frac{1}{2}$ implies that the Rankin–Selberg integral is not identically zero at $s = \frac{1}{2}$, i.e. there is a $\varphi' \in \Pi$ and a Schwartz function $\widetilde{\Phi} \in \mathcal{S}(\mathbb{A}_n)$ with

$$Z\left(\frac{1}{2}, \varphi', \widetilde{\Phi}\right) \neq 0.$$

Moreover, since Π is the weak base change of π, by [Zha, Theorem 1.2], it is distinguished (resp. η-distinguished) if n is odd (resp. even). This means there is an $\varphi \in \Pi$ with

$$\mathcal{P}(\varphi) \neq 0.$$

Let $\widetilde{F} \in \mathcal{S}(\mathsf{G}(\mathbb{A}'))$ be a function such that $\Pi(\widetilde{F})\varphi = \varphi'$ and $\Pi(\widetilde{F})\varphi'' = 0$ for all φ'' orthogonal to φ. Then

$$\mathscr{I}_\Pi(\widetilde{F}, \widetilde{\Phi}) = Z\left(\frac{1}{2}, \Pi(\widetilde{F})\varphi, \widetilde{\Phi}\right) \mathcal{P}(\varphi) \neq 0.$$

We may assume further that $\widetilde{F}$ and $\widetilde{\Phi}$ are factorizable.

Let v be an infinite place. By the theory of Archimedean Rankin–Selberg convolution [Jac2009, Theorem 2.3], $Z_v^*\left(\Pi(\widetilde{F}_v)\varphi_v, \widetilde{\Phi}_v\right)$ is a continuous functional on $\widetilde{\Phi}_v$. This implies that there is a Schwartz function

$$\Phi_v = \sum_{j=1}^r \phi_{1,v}^{(j)} \otimes \phi_{2,v}^{(j)},$$

such that

$$Z_v^*\left(\Pi(\widetilde{F}_v)\varphi_v, \Phi_v\right) \neq 0.$$

We can even take

$$\Phi_v = \phi_{1,v} \otimes \phi_{2,v}.$$

At any nonarchimedean split place v, one knows that

$$\mathscr{I}_{\Pi_v}(F_v, \Phi_v) = \mathscr{J}_{\pi_v}(f_v, \phi_v),$$

if Π_v is a local base change of π_v, and the test functions (F_v, Φ_v) and (f_v, ϕ_v) are related by Proposition 5.3.1. This implies that at any split non-archimedean place v, the distribution $\mathscr{J}_{\pi_v}$ is nonzero. In particular, by Proposition 6.1.2 and Corollary 5.3.2, at the place v_2, we can choose the test function (F_{v_2}, Φ_{v_2}) which is regularly supported and

$$\mathscr{I}_{\Pi_{v_2}}(F_{v_2}, \Phi_{v_2}) \neq 0.$$

The same argument applies to the place v_1. Since $\ell_{v_1} \neq 0$, we can choose a test function (f_{v_1}, ϕ_{v_1}) with f_{v_1} a truncated matrix coefficient, such that

$$\mathscr{J}_{\pi_{v_1}}(f_{v_1}, \phi_{v_1}) \neq 0.$$

Let (F_{v_1}, Φ_{v_1}) be a smooth transfer of (f_{v_1}, ϕ_{v_1}). Then F_{v_1} is a truncated matrix coefficient and

$$\mathscr{I}_{\Pi_{v_1}}(F_{v_1}, \Phi_{v_1}) \neq 0.$$

We now modify $(\widetilde{F}, \widetilde{\Phi})$ at the infinite places and the split places v_1, v_2 by the new choices of test functions as above. Let $\{(f^V, \phi^V)\}_{V \in \mathsf{Herm}_n(k)}$ be the smooth transfer of the new test function (F, Φ). All the test functions are good. So we can apply Proposition 5.6.1. Since the left hand side of identity (5.6.5) is nonzero, there is at least one nonzero term on the right hand side of (5.6.5). This term gives a nonzero Fourier–Jacobi period.

7. Proof of Proposition 6.1.2

7.1. **Setup.** In this section, we prove Proposition 6.1.2. In fact, we shall prove a slightly general Theorem 7.1.1 below. We are always in the local situation from now on, so we drop all the subscripts v. We change our notation slightly from the previous sections.

Let k be an non-archimedean local field (this is our k' in the previous sections), $G = \mathrm{GL}_n(k) \times \mathrm{GL}_n(k)$ and $H = \mathrm{GL}_n(k)$ diagonally embedded in G. Let Z be the center of G, B be the standard Borel subgroup, M the diagonal torus, M^+ the subset of M defined by

$$\{m \in M \mid |\alpha(m)| \leq 1 \text{ for all positive root } \alpha \text{ of } G\}.$$

Let π be an irreducible admissible representation of G, $\widetilde{\pi}$ its contragradient representation. We assume that π is unitary with an invariant hermitian pairing $\langle -, - \rangle$, and identify $\widetilde{\pi}$ with $\overline{\pi}$.

Let $\mu : k^\times \to \mathbb{C}^\times$ be a multiplicative character. Let ω_μ the Weil representation of H, i.e.

$$\omega_\mu(h)\phi(x) = \mu(\det h)|\det h|^{\frac{1}{2}}\phi(xh).$$

To simplify notation, we define the action $\phi^h(x) = \omega_\mu(h)\phi(x)$. There is a natural nondegenerate pairing between ω_μ and $\overline{\omega_\mu}$

$$\langle \phi_1, \phi_2 \rangle = \int_{k_n} \phi_1(x)\overline{\phi_2(x)}\mathrm{d}x.$$

This pairing is invariant under the action of ω_μ.

Recall that for $f \in \mathcal{S}(G)$ and $\phi_1, \phi_2 \in \mathcal{S}(k_n)$,

(7.1.1) $$\mathscr{I}_\pi(f, \phi_1 \otimes \phi_2) = \sum_{\varphi \in \pi} \ell(\pi(f)\varphi, \phi_1)\overline{\ell(\varphi, \phi_2)},$$

where ℓ is a generator of

$$\mathrm{Hom}_H(\pi \otimes \overline{\omega_\mu}, \mathbb{C}).$$

We have introduced the following partial Fourier transform. Let ψ be a non-trivial additive character of k, $\phi_1, \phi_2 \in \mathcal{S}(k_n)$ two Schwartz functions on k_n. Define the partial Fourier transform

$$-^\dagger : \mathcal{S}(k_n) \otimes \mathcal{S}(k_n) \to \mathcal{S}(k_n) \otimes \mathcal{S}(k_n)$$

$$(\phi_1 \otimes \phi_2)^\dagger (x, y) = \int_{k_n} \phi_1(x + z)\phi_2(x - z)\psi(z\,^t y)\mathrm{d}z.$$

For $h \in H$, one has

(7.1.2) $$\left(\phi_1^h \otimes \phi_2^h\right)^\dagger (x, y) = \mu(\det h)^2 \left(\phi_1 \otimes \phi_2\right)^\dagger (xh, y\,^t h^{-1}).$$

Our goal of this section is to prove the following

Theorem 7.1.1. *Assume that π is supercuspidal. Assume that the distribution $\mathscr{I}_\pi$ is nontrivial. Let U be any Zariski open dense subset of $G \times k_n \times k_n$. Then there is a test function*

$$(f, \phi),$$

with $f \in \mathcal{S}(G)$, $\phi = \sum \phi_j \otimes \phi_j'$, and $\phi_j, \phi_j' \in \mathcal{S}(k_n)$, such that
  (1) *$\mathscr{I}_\pi(f, \phi) \neq 0$.*
  (2) *The function on $G \times k_n \times k_n$*

$$(g, x, y) \mapsto f(g) \left(\sum_j \phi_j^{g_1^{-1}} \otimes \phi_j'\right)^\dagger (x, y)$$

  is supported in U. Here $g = (g_1, g_2) \in G$.

The regular semisimple locus of $\mathrm{GL}_n(k) \times \mathrm{GL}_n(k) \times k_n \times k_n$ is an open dense subset. Thus Theorem 7.1.1 does imply Proposition 6.1.2.

7.2. Integration of matrix coefficients. The first step towards Theorem 7.1.1 is to establish an analogue of a theorem of Sakellaridis–Venkatesh [SV, Proposition 6.4.1], [IZ, Theorem A.1]. Let $\varphi \in \pi, \widetilde{\varphi} \in \widetilde{\pi}$ and $\phi_1, \phi_2 \in \mathcal{S}(k_n)$. Consider the following matrix coefficient integration

$$(7.2.1) \qquad \int_H \langle \pi(h)\varphi, \widetilde{\varphi} \rangle \langle \phi_1^h, \phi_2 \rangle \mathrm{d}h.$$

When the integral is absolutely convergent, it gives an element in

$$\mathrm{Hom}_{H \times H}(\pi \otimes \widetilde{\pi} \otimes \omega_\mu \otimes \overline{\omega_\mu}, \mathbb{C}).$$

Theorem 7.2.1. *Assume that π is tempered. Then the matrix coefficient integration (7.2.1) is absolutely convergent. If further more π is supercuspidal, then it is not identically zero.*

We first show the absolute convergence. Let us introduce the Harish-Chandra function Ξ. Let $K = G(\mathfrak{o}_k)$ be a hyperspecial maximal compact subgroup of $G(k)$. The Harish-Chandra function is defined to be

$$\Xi(g) = \int_K \kappa(kg)\mathrm{d}k.$$

where $\kappa \in \mathrm{Ind}_B^G 1$ and its restriction to K is identically 1. Let $\|\cdot\| : G \to \mathbb{R}_{\geq 0}$ be any height function on G, e.g.

$$\|h\| = \sup \left\{ 1, \sum_{i,j} |h_{ij}| + |\det h^{-1}| \right\},$$

and define $\sigma(h) = \log\|h\|$. A function f on G is said to satisfy the *weak inequality* if there are some constants C and C', such that

$$|f(g)| \leq C \times \Xi(g)(1 + \sigma(g))^{C'}.$$

A matrix coefficient of a tempered representation satisfies the weak inequality.

Let δ_G be the modulus function of the standard Borel subgroup of G. For any $m \in M^+$, one has the estimate

$$A^{-1}\delta(m)^{\frac{1}{2}} \leq \Xi(m) \leq A\delta(m)^{\frac{1}{2}}(1 + \sigma(m))^B$$

for some constants A and B. The function Ξ is bi-KZ-invariant, and $\Xi(g) = \Xi(g^{-1})$.

Since π is supercuspidal, it is tempered. Thus to prove the absolute convergence of (7.2.1), we only need to show that

Proposition 7.2.2. *The integral*

$$(7.2.2) \qquad \int_H \Xi(h)(1 + \sigma(h))^{C'} \left| \int_{k_n} \phi_1^h(x)\phi_2(x)\mathrm{d}x \right| \mathrm{d}h$$

is absolutely convergent for any $\phi_1, \phi_2 \in \mathcal{S}(k_n)$ and any constant C'.

Since ϕ is compactly supported, there is an N such that

$$|\phi(x)| \leq \sup \phi \times \mathbf{1}_{\varpi^{-N}\mathfrak{o}_n}(x).$$

Thus by making suitable change of variables, we are reduced to show

Lemma 7.2.3. *The integral*

$$(7.2.3) \qquad \int_H \Xi(h)(1 + \sigma(h))^{C'} \left| \int_{k_n} \mathbf{1}_{\mathfrak{o}_n}^h(x)\, \mathbf{1}_{\mathfrak{o}_n}(x)\mathrm{d}x \right| \mathrm{d}h$$

is absolutely convergent for any constant C'.

Proof. The Cartan decomposition gives

$$\mathrm{GL}_n(k) = \coprod_{j_1 \geq \cdots \geq j_n} K_{j_1, \cdots, j_n} := \coprod_{j_1 \geq \cdots \geq j_n} \mathrm{GL}_n(\mathfrak{o}) \operatorname{diag}[\varpi^{j_1}, \cdots, \varpi^{j_n}] \mathrm{GL}_n(\mathfrak{o}).$$

Thus

$$
(7.2.4) \quad
\begin{aligned}
(7.2.3) = \sum_{j_1 \geq \cdots \geq j_n} &\Xi(\operatorname{diag}[\varpi^{j_1}, \cdots, \varpi^{j_n}])(1 + \log q \cdot \max(|j_1|, \cdots, |j_n|) \operatorname{vol} K_{j_1, \cdots, j_n} \\
&q^{\frac{1}{2}(j_1 + \cdots + j_n)} \operatorname{vol}(\mathfrak{o}_n \operatorname{diag}[\varpi^{-j_1}, \cdots, \varpi^{-j_n}] \cap \mathfrak{o}_n).
\end{aligned}
$$

By [II2010, page 1388], one has

$$A^{-1} \delta_H^{-1}(\operatorname{diag}[\varpi^{-j_1}, \cdots, \varpi^{-j_n}]) \leq \operatorname{vol} K_{j_1, \cdots, j_n} \leq A \delta_H^{-1}(\operatorname{diag}[\varpi^{-j_1}, \cdots, \varpi^{-j_n}]).$$

Then we are reduced to show the convergence of

$$\sum_{r=0}^{n} \sum_{\substack{j_1 \geq \cdots \geq j_r \geq 0 \\ 0 \geq j_{r+1} \geq \cdots \geq j_n}} q^{-\frac{1}{2}(j_1 + \cdots + j_r) + \frac{1}{2}(j_{r+1} + \cdots + j_n)}(1 + \log q \cdot \max(|j_1|, \cdots, |j_n|))^B$$

for any constant B. But this is clear. $\qquad\square$

We now assume that π is supercuspidal and prove that (7.2.1) is not identically zero.

Let $\pi = \sigma \boxtimes \tau$ where σ and τ are two supercuspidal representations of $\mathrm{GL}_n(k)$. Consider the Weil representation of $\mathrm{GL}_m \times \mathrm{GL}_n$ as defined in Section 2.1. Let $\mathcal{S}(\mathrm{Mat}_{m,n})$ be the space of Schwartz functions on $\mathrm{Mat}_{m,n}$. The Weil representation of $\mathrm{GL}_n(k) \times \mathrm{GL}_m(k)$ is realized on $\mathcal{S}(\mathrm{Mat}_{m,n})$ as

$$\omega_\mu(g, h)\Phi(X) = \mu(\det g)^{-1} \mu(\det h) |\det g|^{-\frac{n}{2}} |\det h|^{\frac{m}{2}} \phi(g^{-1} X h),$$

for $g \in \mathrm{GL}_m$, $h \in \mathrm{GL}_n$ and $\Phi \in \mathcal{S}(\mathrm{Mat}_{m,n})$. One special case of this action is important for us. Let $m = n + 1$. There is an embedding, given by

$$\iota : \mathrm{GL}_n \to \mathrm{GL}_{n+1}, \quad h \mapsto \operatorname{diag}[h, 1].$$

Then

$$\mathcal{S}(\mathrm{Mat}_{n+1,n}) \simeq \mathcal{S}(\mathrm{Mat}_{n,n}) \otimes \mathcal{S}(k_n).$$

as the representation of $\iota(GL_n(k)) \times \mathrm{GL}_n(k)$. Explicitly, let

$$\Phi = \Psi \otimes \phi \in \mathcal{S}(\mathrm{Mat}_{n,n}) \otimes \mathcal{S}(k_n).$$

Then

$$\omega_\mu(\iota(h_1), h_2)(\Psi \otimes \phi)(X, x) = \omega_\mu(h_1, h_2)\Psi(X)\omega_\mu(h_2)\phi(x).$$

Let $\varphi_\sigma \in \sigma$, $\widetilde{\varphi_\sigma} \in \widetilde{\sigma}$, $\varphi_\tau \in \tau$, $\widetilde{\varphi_\tau} \in \widetilde{\tau}$, $\Phi, \widetilde{\Phi} \in \mathcal{S}(\mathrm{Mat}_{n+1,n})$. Consider the following integral

$$(7.2.5) \quad \iint_{(\mathrm{GL}_n(k))^2} \langle \tau(h_2)\varphi_\tau, \widetilde{\varphi_\tau} \rangle \langle \sigma(\iota(h_1))\varphi_\sigma, \widetilde{\varphi_\sigma} \rangle \langle \omega_\mu(\iota(h_1), h_2)\Phi, \widetilde{\Phi} \rangle \mathrm{d}h_1 \mathrm{d}h_2.$$

Lemma 7.2.4. *This integral is absolutely convergent.*

Proof. Since π is supercuspidal, one only need to prove that the integral over the center is absolutely convergent. This is similar to (and in fact simpler than) Proposition 7.2.2. $\qquad\square$

According to the explicit theta lift of Gan and Ichino [GI2014, § 16], the integral

$$\int_{\mathrm{GL}_n} \langle \tau(h_2)\varphi_\sigma, \widetilde{\varphi_\sigma} \rangle \langle \omega_\mu(g, h_2)\Phi, \widetilde{\Phi} \rangle \mathrm{d}h_2$$

is a matrix coefficient of $\theta(\sigma)$ where $\theta(\sigma)$ stands for the theta lifting of σ to GL_{n+1}. In fact, all the matrix coefficients of $\theta(\sigma)$ arise in this way. Since

$$\dim \mathrm{Hom}_H(\pi \otimes \overline{\omega_\mu}, \mathbb{C}) = 1,$$

one deduces as in [GGP2012, Theorem 14.1] that

$$\dim \mathrm{Hom}_H(\theta(\sigma) \otimes \tau, \mathbb{C}) = 1.$$

The theta lifting $\theta(\sigma)$ is thus nonzero. It is irreducible since σ is a supercuspidal representation. It is tempered since σ is tempered. Thus by the theorem of Sakellaridis–Venkatesh, there is a choice of the test functions φ_τ, $\widetilde{\varphi_\tau}$ etc., such that the integral (7.2.5) is nonzero. Note that any Φ is a finite sum of the functions of the form

$$\Psi \otimes \phi \in \mathcal{S}(\mathrm{Mat}_{n,n}) \otimes \mathcal{S}(k_n).$$

One can actually choose $\Phi = \Psi \otimes \phi, \widetilde{\Phi} = \widetilde{\Psi} \otimes \widetilde{\phi}$, such that (7.2.5) is not zero.

Now consider the integral

$$\int_{\mathrm{GL}_n} \langle \omega_\mu(h_1, h_2)\Psi, \widetilde{\Psi} \rangle \langle \tau(h_1)\varphi_\tau, \widetilde{\varphi_\tau} \rangle \mathrm{d}h_1.$$

This is a matrix coefficient of $\theta(\tau)$, the theta lifting of τ to GL_n, which is nothing but the contragradient of τ. Thus the integral (7.2.5) takes the form (7.2.1), and is not zero with our choices of φ_σ, etc. The theorem is thus proved.

7.3. Representability of the distribution.

Assume from now on that π is supercupspidal and $\mathscr{J} \neq 0$. In order to simplify the notation, we assume that μ is trivial character. The general case can be treated in exactly the same way. For the rest of this section, we are going to follow the argument in [IZ].

From Theorem 7.2.1, we see that the generator ℓ can be taken to be a matrix coefficient integration. Fix such a choice, i.e. an $\varphi_0 \in \pi$ and $\phi_0 \in \mathcal{S}(k_n)$ such that (7.2.1) does not vanish identically and

$$\ell(\varphi, \phi) = \int_H \langle \pi(h)\varphi, \varphi_0 \rangle \langle \phi^h, \phi_0 \rangle \mathrm{d}h.$$

Then we have

Lemma 7.3.1.

$$\mathscr{J}_\pi(f, \phi_1 \otimes \phi_2) = \iint_{H^2} \left(\int_G f(g)\langle \pi(h_1 g h_2^{-1})\varphi_0, \varphi_0 \rangle \mathrm{d}g \right)$$
$$\left(\iint_{(k_n)^2} \left(\phi_1^{g_1^{-1}} \otimes \overline{\phi_2} \right)^\dagger (x,y) \overline{\left(\phi_0^{g_1^{-1}h_1^{-1}} \otimes \overline{\phi_0^{h_2^{-1}}} \right)^\dagger (x,y)} \mathrm{d}x\mathrm{d}y \right) \mathrm{d}h_1 \mathrm{d}h_2.$$

Proof. Proceeding exactly as in [IZ, Lemma A.3], one obtains that

$$\mathscr{J}_\pi(f, \phi_1 \otimes \phi_2) = \iint_{H^2} \left(\int_G f(g)\langle \pi(h_1 g h_2^{-1})\varphi_0, \varphi_0 \rangle \mathrm{d}g \right)$$
$$\left(\iint_{(k_n)^2} \phi_1^{g_1^{-1}}(x) \overline{\phi_0^{g_1^{-1}h_1^{-1}}}(x) \overline{\phi_2}(y) \phi_0^{h_2^{-1}}(y) \mathrm{d}x\mathrm{d}y \right) \mathrm{d}h_1 \mathrm{d}h_2.$$

Since Fourier transform preserves L^2-norm, one has

(7.3.1)
$$\iint_{(k_n)^2} \phi_1^{g_1^{-1}}(x) \overline{\phi_0^{g_1^{-1}h_1^{-1}}}(x) \overline{\phi_2}(y) \phi_0^{h_2^{-1}}(y) \mathrm{d}x\mathrm{d}y$$
$$= \iint_{(k_n)^2} \left(\phi_1^{g_1^{-1}} \otimes \overline{\phi_2} \right)^\dagger (x,y) \overline{\left(\phi_0^{g_1^{-1}h_1^{-1}} \otimes \overline{\phi_0^{h_2^{-1}}} \right)^\dagger (x,y)} \mathrm{d}x\mathrm{d}y.$$

The lemma then follows. $\square$

Let m_1 (resp. m_2) be a matrix coefficient of σ (resp. τ). They are smooth functions on $\mathrm{GL}_n(k)$ that are compactly supported modulo center. Let ϕ_1 and ϕ_2 be two Schwartz function on k_n.

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Lemma 7.3.2. *The function*

$$(g_1, g_2, x, y) \mapsto \iint\limits_{H^2} \left| m_1(h_1 g_1 h_2^{-1}) m_2(h_1 g_2 h_2^{-1}) \left(\phi_1^{g_1^{-1} h_1^{-1}} \otimes \phi_2^{h_1^{-1}} \right)^\dagger (x, y) \right| \mathrm{d}h_1 \mathrm{d}h_2$$

is locally integrable on $G \times k_n \times k_n$.

Proof. We first make the change of variable $h_1 \mapsto h_1 h_2 g_1^{-1}$. The integral in question becomes

$$(7.3.2) \qquad \iint\limits_{H^2} \left| m_1(h_1) m_2(h_1 h_2 g_1^{-1} g_2 h_2) \left(\phi_1^{h_1^{-1}} \otimes \phi_2 \right)^\dagger (x h_2, y\,{}^t h_2^{-1}) \right| \mathrm{d}h_1 \mathrm{d}h_2.$$

Since m_1 is compactly supported modulo center, we are reduced to show that for any function m on GL_n that is compactly supported modulo center, as a function of $(g, x, y) \in \mathrm{GL}_n(k) \times k_n \times k_n$,

$$\int\limits_{k^\times} \int\limits_{H} \left| m(h g h^{-1}) (\phi_1^z \otimes \phi_2)^\dagger (x h, y\,{}^t h^{-1}) \right| \mathrm{d}h \mathrm{d}z$$

is locally integrable. By writing h as $k_1 t k_2$ where $k_1, k_2 \in \mathrm{GL}_n(\mathfrak{o})$ we are then reduced to show that

$$(7.3.3) \qquad \int\limits_{k^\times} \int\limits_{M(k)} \left| m(h g h^{-1}) (\phi_1^z \otimes \phi_2)^\dagger (x h, y\,{}^t h^{-1}) \right| \mathrm{d}h \mathrm{d}z$$

is locally integrable where M is the diagonal torus.

By writing ϕ_1 and ϕ_2 as a linear combination of the functions of the form $\mathbf{1}_{a + \varpi^M \mathfrak{o}_n}$ for some $a \in k^\times$ and integer M, we are further reduced to show the integral (7.3.3) is locally integrable if $\phi_1 = \mathbf{1}_{a + \varpi^M \mathfrak{o}_n}$ and $\phi_2 = \mathbf{1}_{b + \varpi^M \mathfrak{o}_n}$.

Direct computation shows

$$\left| (\mathbf{1}_{a + \varpi^M \mathfrak{o}_n}^z \otimes \mathbf{1}_{b + \varpi^M \mathfrak{o}_n})^\dagger (x, y) \right| = \begin{cases} \mathbf{1}_{\varpi^M \mathfrak{o}_n} \left(x - \dfrac{a z^{-1} + b}{2} \right) \mathbf{1}_{z \varpi^{-M} \operatorname{cond} \psi \mathfrak{o}_n}(y) & |z| \geq 1 \\[3mm] \mathbf{1}_{z^{-1} \varpi^M \mathfrak{o}_n} \left(x - \dfrac{a z^{-1} + b}{2} \right) \mathbf{1}_{\varpi^{-M} \mathfrak{o}_n \operatorname{cond} \psi}(y) & |z| < 1. \end{cases}$$

We also note that for any $R \in k$ and any integers N, N',

$$\int\limits_{k} \mathbf{1}_{\varpi^N \mathfrak{o}}(x - R)\, \mathbf{1}_{\varpi^{N'} \mathfrak{o}}(x) \mathrm{d}x \leq \int\limits_{k} \mathbf{1}_{\varpi^N \mathfrak{o}}(x)\, \mathbf{1}_{\varpi^{N'} \mathfrak{o}}(x) \mathrm{d}x.$$

Therefore to show that (7.3.3) is locally integrable, we only need to show that for any integer N and any compact subset K of $\mathrm{GL}_n(k)$, the integrals

$$\iint\limits_{(k_n)^2} \int\limits_{K} \int\limits_{|z| \geq 1} \int\limits_{M(k)} |m(h^{-1} g h)|\, \mathbf{1}_{\varpi^M \mathfrak{o}_n}(x h)\, \mathbf{1}_{z \varpi^{-M} \operatorname{cond} \psi \mathfrak{o}_n}(y h^{-1})\, \mathbf{1}_{\varpi^N \mathfrak{o}_n}(x)\, \mathbf{1}_{\varpi^N \mathfrak{o}_n}(y) \mathrm{d}h \mathrm{d}z \mathrm{d}g \mathrm{d}x \mathrm{d}y$$

and

$$\iint\limits_{(k_n)^2} \int\limits_{K} \int\limits_{|z| < 1} \int\limits_{M(k)} |m(h^{-1} g h)|\, \mathbf{1}_{z^{-1} \varpi^M \mathfrak{o}_n}(x h)\, \mathbf{1}_{\varpi^{-M} \operatorname{cond} \psi \mathfrak{o}_n}(y h^{-1})\, \mathbf{1}_{\varpi^N \mathfrak{o}_n}(x)\, \mathbf{1}_{\varpi^N \mathfrak{o}_n}(y) \mathrm{d}h \mathrm{d}z \mathrm{d}g \mathrm{d}x \mathrm{d}y$$

are absolutely convergent. The same argument as the proof of Proposition 7.2.2 shows that these integrals are absolutely convergent. The lemma is thus proved. $\qquad \square$

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Now for $\varphi, \widetilde{\varphi} \in \pi$ and a Schwartz function $\phi \in \mathcal{S}(k_n)$, we define the functions $\mathbf{O}_{\varphi,\widetilde{\varphi},\phi}(g,x,y)$ and $\mathbf{O}_{|\varphi,\widetilde{\varphi},\phi|}(g,x,y)$ on $G \times k_n \times k_n$ as

$$\mathbf{O}_{\varphi,\widetilde{\varphi},\phi}(g,x,y) = \iint\limits_{H^2} \langle \pi(h_1 g h_2^{-1})\varphi, \widetilde{\varphi}\rangle \overline{\left(\phi^{g_1^{-1}h_1^{-1}} \otimes \overline{\phi^{h_2^{-1}}}\right)^\dagger}(x,y) \mathrm{d}h_1 \mathrm{d}h_2,$$

$$\mathbf{O}_{|\varphi,\widetilde{\varphi},\phi|}(g,x,y) = \iint\limits_{H^2} \left| \langle \pi(h_1 g h_2^{-1})\varphi, \widetilde{\varphi}\rangle \overline{\left(\phi^{g_1^{-1}h_1^{-1}} \otimes \overline{\phi^{h_2^{-1}}}\right)^\dagger}(x,y)\right| \mathrm{d}h_1 \mathrm{d}h_2.$$

These functions are locally integrable by the lemma above.

Theorem 7.3.3. *For any function $f \in \mathcal{S}(G)$ and any Schwartz functions $\phi_1, \phi_2 \in \mathcal{S}(k_n)$, the integral*

$$\int\limits_{G} \iint\limits_{(k_n)^2} f(g_1,g_2) \left(\phi_1^{g_1^{-1}} \otimes \overline{\phi_2}\right)^\dagger (x,y) \mathbf{O}_{\varphi_0,\varphi_0,\phi_0}(g,x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}g$$

is absolutely convergent. Here we write $g = (g_1, g_2)$. Moreover it equals $\mathscr{I}_\pi(f, \phi_1 \otimes \phi_2)$.

Proof. The absolute convergence is proved in the previous lemma.
  Let

$$\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_m \subset \cdots \subset H, \quad \bigcup_m \Omega_m = H$$

be an increasing sequence of open compact subset of H. Define

$$\mathbf{O}^{(m)}_{\varphi_0,\varphi_0,\phi_0}(g,x,y) = \iint\limits_{(\Omega_m)^2} \langle \pi(h_1 g h_2^{-1})\varphi_0, \varphi_0\rangle \overline{\left(\phi_0^{g_1^{-1}h_1^{-1}} \otimes \overline{\phi_0^{h_2^{-1}}}\right)^\dagger}(x,y)\mathrm{d}h_1\mathrm{d}h_2.$$

Similarly we define $\mathbf{O}^{(m)}_{|\varphi_0,\varphi_0,\phi_0|}(g,x,y)$ They pointwisely

$$|\mathbf{O}^{(m)}_{\varphi_0,\varphi_0,\phi_0}| \leq \mathbf{O}^{(m)}_{|\varphi_0,\varphi_0,\phi_0|}.$$

The function $\mathbf{O}^{(m)}_{|\varphi_0,\varphi_0,\phi_0|}$ is locally integrable by Lemma 7.3.2. Therefore by Lebesgue dominant convergence theorem, one has

$$\lim_{m\to\infty} \int\limits_{G} \iint\limits_{(k_n)^2} f(g) \left(\phi_1^{g_1^{-1}} \otimes \phi_2\right)^\dagger (x,y) \mathbf{O}^{(m)}_{\varphi_0,\varphi_0,\phi_0}(g,x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}g$$

$$= \int\limits_{G} \iint\limits_{(k_n)^2} f(g) \left(\phi_1^{g_1^{-1}} \otimes \phi_2\right)^\dagger (x,y) \lim_{m\to\infty} \mathbf{O}^{(n)}_{\varphi_0,\varphi_0,\phi_0}(g,x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}g$$

$$= \int\limits_{G} \iint\limits_{(k_n)^2} f(g) \left(\phi_1^{g_1^{-1}} \otimes \phi_2\right)^\dagger (x,y) \mathbf{O}_{\varphi_0,\varphi_0,\phi_0}(g,x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}g.$$

On the other hand, since the support of $f(g) \left(\phi_1^{g_1^{-1}} \otimes \phi_2\right)^\dagger (x,y)$ as well as the Ω_m's are all compact, we can interchange the order of integration, i.e.

$$\int\limits_{G} \iint\limits_{(k_n)^2} f(g) \left(\phi_1^{g_1^{-1}} \otimes \phi_2\right)^\dagger (x,y) \mathbf{O}^{(m)}_{\varphi_0,\varphi_0,\phi_0}(g,x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}g$$

$$= \iint\limits_{(\Omega_m)^2} \int\limits_{G} \iint\limits_{(k_n)^2} f(g) \left(\phi_1^{g_1^{-1}} \otimes \phi_2\right)^\dagger (x,y) \langle \pi(h_1 g h_2^{-1})\varphi_0, \varphi_0\rangle \overline{\left(\phi_0^{g_1^{-1}h_1^{-1}} \otimes \overline{\phi_0^{h_2^{-1}}}\right)^\dagger}(x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}g\mathrm{d}h_1\mathrm{d}h_2.$$

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Taking $m \to \infty$, we get

$$\lim_{m\to\infty} \int\limits_{G} \iint\limits_{(k_n)^2} f(g) \left(\phi_1^{g_1^{-1}} \otimes \phi_2\right)^\dagger (x,y) \mathbf{O}^{(m)}_{\varphi_0,\varphi_0,\phi_0}(g,x,y) \mathrm{d}x\mathrm{d}y\mathrm{d}g$$

$$= \iint\limits_{H^2} \int\limits_{G} \iint\limits_{(k_n)^2} f(g) \left(\phi_1^{g_1^{-1}} \otimes \phi_2\right)^\dagger (x,y) \langle \pi(h_1 g h_2^{-1})\varphi_0,\varphi_0\rangle \overline{\left(\phi_0^{g_1^{-1} h_1^{-1}} \otimes \overline{\phi_2^{h_2^{-1}}}\right)^\dagger} (x,y) \mathrm{d}x\mathrm{d}y\mathrm{d}g\mathrm{d}h_1\mathrm{d}h_2.$$

This proves our theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7.4. **Proof of Theorem 7.1.1.** Now we can prove our main theorem. By assumption, there is an f and $\phi_1,\phi_2 \in \mathcal{S}(k_n)$ such that

$$\mathscr{I}_\pi(f,\phi_1 \otimes \phi_2) \neq 0.$$

This in particular implies that there is an f and a Schwartz function on $\phi^\circ \in \mathcal{S}(k_n \times k_n)$ such that

$$\int\limits_{G} \iint\limits_{(k_n)^2} f(g)\phi^\circ(x,y) \mathbf{O}_{\varphi_0,\varphi_0,\phi_0}(g,x,y) \mathrm{d}x\mathrm{d}y\mathrm{d}g \neq 0.$$

For simplicity, we may assume that both f and ϕ are nonnegative. Since U is open dense in $G \times k_n \times k_n$, one can choose a sequence of functions $f_i \otimes \phi_i^\circ \in \mathcal{S}(U)$ such that pointwisely on U, one has

$$\lim_{i\to\infty} f_i(g)\phi_i^\circ(x,y) = f(g)\phi^\circ(x,y).$$

Moreover, one may assume that $f_i \leq f$ and $\phi_i^\circ \leq \phi^\circ$.

With this choice, one has

$$|f(g)\phi^\circ(x,y) - f_i(g)\phi_i^\circ(x,y)| \leq 2f(g)\phi^\circ(x,y).$$

Therefore by Lebesgue dominant convergence theorem, one obtains

$$\lim_{i\to\infty} \int\limits_{G} \iint\limits_{(k_n)^2} f_i(g)\phi_i^\circ(x,y) \mathbf{O}_{\varphi_0,\varphi_0,\phi_0}(g,x,y) \mathrm{d}x\mathrm{d}y\mathrm{d}g = \int\limits_{G} \iint\limits_{(k_n)^2} f(g)\phi^\circ(x,y) \mathbf{O}_{\varphi_0,\varphi_0,\phi_0}(g,x,y) \mathrm{d}x\mathrm{d}y\mathrm{d}g.$$

Thus for some i sufficiently large, one must have

$$\int\limits_{G} \iint\limits_{(k_n)^2} f_i(g)\phi_i^\circ(x,y) \mathbf{O}_{\varphi_0,\varphi_0,\phi_0}(g,x,y) \mathrm{d}x\mathrm{d}y\mathrm{d}g \neq 0.$$

Moreover $f_i \otimes \phi_i^\circ$ is supported in U.

Let $\widetilde{\phi}$ be the inverse Fourier transform of ϕ_i°. Then $\widetilde{\phi}$ takes the form

$$\sum_{j=1}^r \phi_j \otimes \phi_j'.$$

Let K be the support of f_i. This is an open compact subset of G. Then there is a finite partition of K

$$K = \bigcup_\alpha K_\alpha,$$

with the properties that

    (1) Each K_i is open compact;
    (2) For any fixed α, the set

$$\left\{ \sum_j \phi_j^{g_1} \otimes \phi_j' \mid (g_1,g_2) \in K_\alpha \right\}$$

    consists of only one element.

For each α, denote the element in the above set by ϕ_α.

Thus we have

$$\sum_\alpha \int_G \iint_{(k_n)^2} f_i(g)\,\mathbf{1}_{K_\alpha}(g)\left(\sum_j (\phi_j^{g_1})^{g_1^{-1}} \otimes \phi_j'\right)^\dagger (x,y)\mathbf{O}_{\varphi_0,\varphi_0,\phi_0}(g,x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}g \neq 0.$$

Therefore for some α, the integral is not zero. Let $f(g) = f_i(g)\,\mathbf{1}_{K_\alpha}(g)$, and $\phi = \phi_\alpha$. We thus have proved Theorem 7.1.1.

8. Application to the Bessel period

8.1. **The theta correspondences.** In this section, we apply the results on the GGP conjecture for $\mathrm{U}(n) \times \mathrm{U}(n)$ to the GGP conjecture for $\mathrm{U}(n+1) \times \mathrm{U}(n)$ via theta correspondences.

First we recall some basic results of the theta correspondence, following the work of Howe [How1989], Kudla [Kud1986], Moeglin–Vigneras–Waldspurger [MVW1987], Waldspurger [Wal1990] and the recent work of Gan–Takeda [GT]. Assume we are in the local situation. We keep the notation from Section 2.1. Let W and V be two Hermitian spaces over k. There is a Weil representation $\omega_{\psi,\boldsymbol{\mu},W,V}$ of $\mathrm{U}(W)\times\mathrm{U}(V)$. To simplify notation, we shall drop the characters from the notations when there is no confusion. For any irreducible admissible representation π of $\mathrm{U}(W)(k')$, the maximal π-isotypic component of $\omega_{W,V}$ takes the form

$$\pi \boxtimes \Theta_{\psi,\boldsymbol{\mu},W,V}(\pi).$$

It is known that $\Theta_{\psi,\boldsymbol{\mu},W,V}(\pi)$ is an admissible representation of $\mathrm{U}(V)(k')$ of finite length. Let $\theta_{\psi,\boldsymbol{\mu},W,V}(\pi)$ be the maximal semisimple quotient of $\Theta_{\psi,\boldsymbol{\mu},W,V}(\pi)$. The Howe duality conjecture claims: the representation $\theta_{\psi,\boldsymbol{\mu},W,V}(\pi)$ is irreducible and $\pi \simeq \pi'$ if and only if $\theta_{\psi,\boldsymbol{\mu},W,V}(\pi) \simeq \theta_{\psi,\boldsymbol{\mu},W,V}(\pi')$. Howe [How1989] proved the conjecture when k' is archimedean and Waldspurger [Wal1990] proved it when k' is local and the residue characteristic of k' is not two. Gan and Takeda [GT] has recently proved that if $|\dim W - \dim V| \leq 1$, then Howe's duality conjecture holds even if the residue characteristic of k' is two. We are only going to use theta correspondence when $|\dim W - \dim V| \leq 1$ in the following.

Now assume that k/k' is a quadratic extension of global fields and we are in the global situation as defined in Section 1.2. We keep to notation of Section 2.2. Let V and W be two Hermitian spaces over k. Recall that we have defined the symplectic space $\mathrm{Res}_{k/k'}(V \otimes W)$ in Section 2.1. Let $\mathbf{L} \subset \mathrm{Res}_{k/k'}(V \otimes W)$ be a Lagrangian subspace. The Weil representation $\omega_{\psi,\boldsymbol{\mu},V,W}$ is realized on the space of Schwartz functions $\mathcal{S}(\mathbf{L}(\mathbb{A}'))$. The theta function is defined by

$$\theta_{\psi,\boldsymbol{\mu},W,V}(g,h,\phi) = \sum_{x \in \mathbf{L}(k')} \omega_{\psi,\boldsymbol{\mu},W,V}(g,h)\phi(x),$$

where $g \in \mathrm{U}(W)(\mathbb{A}')$, $h \in \mathrm{U}(V)(\mathbb{A}')$ and $\phi \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$. Let π be an irreducible cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}')$. The theta lift $\theta_{\psi,\boldsymbol{\mu},W,V}(\pi)$ of π is defined to be the span of the functions on $\mathrm{U}(V)(\mathbb{A}')$ of the form

$$\theta_{\psi,\boldsymbol{\mu},W,V}(\varphi,\phi)(g) = \int_{\mathrm{U}(W)(k')\backslash \mathrm{U}(W)(\mathbb{A}')} \overline{\varphi(h)}\theta_{\psi,\boldsymbol{\mu},W,V}(g,h,\phi)\mathrm{d}h,$$

where $\varphi \in \pi$ and $\phi \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$.

Proposition 8.1.1 ([GRS1993, Proposition 1.2]). *Assume $|\dim W - \dim V| \leq 1$. If $\theta_{\psi,\boldsymbol{\mu},W,V}(\pi)$ is a cuspidal representation of $\mathrm{U}(V)(\mathbb{A}')$, then it is irreducible and is isomorphic to the restricted tensor product $\otimes_v' \theta(\pi_v)$. Moreover, the theta lift of $\theta_{\psi^{-1},\boldsymbol{\mu}^{-1},W,V}(\pi)$ back to $\mathrm{U}(W)(\mathbb{A}')$ is isomorphic to π.*

Remark 8.1.2. The proposition would be true without the condition $|\dim W - \dim V| \leq 1$ if the Howe duality conjecture was proved in complete generality.

To proceed, we need to assume more properties of the weak base change than the previous sections. We make the following hypothesis.

Hypothesis BC: For any irreducible cuspidal automorphic representation π of $\mathrm{U}(W)(\mathbb{A}')$, its weak base change $\mathrm{BC}(\pi)$ is isobaric. Moreover, for any character $\chi : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$, we have

$$L(s, \mathrm{BC}(\pi) \otimes \chi) = L(s, \pi \times \chi),$$

where the right hand side is the L-function defined by the doubling method as in [Yam2014].

One of the most important question in the theory of theta lift is to determine whether $\theta_{\psi, \boldsymbol{\mu}, W, V}(\pi)$ is zero or not. When $\dim W = \dim V$, there is a local-global criterion as follows.

Proposition 8.1.3 ([Yam2014, Theorem 10.1]). *Suppose* $\dim W = \dim V$ *and* π *is an irreducible cuspidal automorphic representation of* $\mathrm{U}(W)(\mathbb{A}')$. *Assume that* $\theta_{\psi, \boldsymbol{\mu}, W, V}(\pi)$ *is a cuspidal automorphic representation of* $\mathrm{U}(V)(\mathbb{A}')$. *Then* $\theta_{\psi, \boldsymbol{\mu}, W, V}(\pi)$ *is not zero if and only if*

(1) *For all the places v of k', the local theta lift $\theta_{\psi_v, \boldsymbol{\mu_v}, W_v, V_v}(\pi_v)$ is not zero;*
(2) $L(1/2, \pi \times \mu^n) \neq 0$.

We need to understand the relation between the weak base change and the theta correspondence. This is a weak version of a conjecture of Prasad [Pra2000].

Proposition 8.1.4. *Assume Hypothesis BC.*

(1) *If* $\dim W = \dim V$, *then* $\mathrm{BC}(\theta_{\psi, \boldsymbol{\mu}, W, V}(\pi)) = \mathrm{BC}(\pi)$.
(2) *If* $\dim W = \dim V - 1$, *then* $\mathrm{BC}(\theta_{\psi, \boldsymbol{\mu}, W, V}(\pi)) = \mu^{-1}\mathrm{BC}(\pi) \boxplus \mu^{\dim W}$.

Proof. By Hypothesis BC, the weak base change is isobaric. Therefore by the strong multiplicity one theorem [JS1981], we only need to check the proposition at the places v of k that is of degree one over k'. The proposition then follows from [Mín2008, Théorème 1]. □

8.2. **GGP conjecture for** $\mathrm{U}(n+1) \times \mathrm{U}(n)$**.** We are now in the global situation. Let V and W be a pair of Hermitian spaces over k of dimension $n+1$ and n respectively and $W \subset V$. Suppose $V = W + E$ where E is an anisotropic line. Let π_V and π_W be irreducible cuspidal automorphic representations of $\mathrm{U}(V)(\mathbb{A}')$ and $\mathrm{U}(W)(\mathbb{A}')$ respectively. We are going to drop the subscripts ψ and $\boldsymbol{\mu}$ for the Weil representation and theta correspondence, and write simply $\omega_{W,V}$ (resp. $\theta_{W,V}$) for $\omega_{\psi, \boldsymbol{\mu}, W, V}$ (resp. $\theta_{\psi, \boldsymbol{\mu}, W, V}$).

The group $\mathrm{U}(W)$ embeds in $\mathrm{U}(V)$ via

$$\iota : \mathrm{U}(W) \to \mathrm{U}(V), \quad g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}.$$

This also gives an embedding $\mathrm{U}(W) \to \mathrm{U}(V) \times \mathrm{U}(W)$.

Let W_0 be another Hermitian space of dimension n. It is worth noting that as representations of $\iota(\mathrm{U}(W)(\mathbb{A}')) \times \mathrm{U}(W_0)(\mathbb{A}')$, $\omega_{V,W_0} = \omega_{W,W_0} \widehat{\otimes} \omega_{W_0}$, where ω_{W_0} is the Weil representation of $\mathrm{U}(W_0)(\mathbb{A}')$ as defined in Section 2.1. Let $\mathbf{L} \subset \mathrm{Res}_{k/k'}(V \otimes W_0)$ be a Lagrangian subspace with the decomposition $\mathbf{L} = \mathbf{L}_n + \mathbf{L}_1$ where $\mathbf{L}_n$ (resp. $\mathbf{L}_1$) is a Lagrangian subspace of $\mathrm{Res}_{k/k'}(W \otimes W_0)$ (resp. $\mathrm{Res}_{k/k'}(W_0)$). Then if $\phi = \phi_n \otimes \phi_1 \in \mathcal{S}(\mathbf{L}_n(\mathbb{A}')) \otimes \mathcal{S}(\mathbf{L}_1(\mathbb{A}'))$, then

$$\theta_{V,W_0}(\iota(g), h, \phi) = \theta_{W,W_0}(g, h, \phi_n)\theta_{W_0}(h, \phi_1).$$

Let π_V (resp. π_W) be an irreducible cuspidal automorphic representation of $\mathrm{U}(V)(\mathbb{A}')$ (resp. $\mathrm{U}(W)(\mathbb{A}')$).

Theorem 8.2.1. *Assume the hypothesis BC as in Section 8.1. Assume the following conditions.*

(1) *All the archimedean places v of k' split in k;*
(2) *There is a Hermitian space W_0, and irreducible cuspidal automorphic representaions π_{W_0} and σ_{W_0} of $\mathrm{U}(W_0)(\mathbb{A}')$, such that π_V is the theta lift of π_0 and π_W is the theta lift of σ_0.*
(3) *There are two split places v_1 and v_2 of k', such that the local components of π_{W_0, v_i} and σ_{W_0, v_i} $(i = 1, 2)$ are all supercuspidal.*

The following are equivalent.

(1) $L(\frac{1}{2}, \mathrm{BC}(\pi_V) \times \mathrm{BC}(\pi_W)) \neq 0$.

(2) *There is a relevant pair $W' \subset V'$ of Hermitian spaces, and an irreducible cuspidal automorphic representation $\pi_{W'}$ (resp. $\pi_{V'}$) of $U(W')(\mathbb{A}')$ (resp. $U(V')(\mathbb{A}')$), which is nearly equivalent to π_W (resp. π_V), such that the period integral*

$$\int_{U(W')(k')\backslash U(W')(\mathbb{A}')} \varphi_{W'}(g)\varphi_{V'}(g)\mathrm{d}g \neq 0,$$

for some $\varphi_{W'} \in \pi_{W'}$ and $\varphi_{V'} \in \pi_{V'}$.

Proof. First of all, let us note the relation between the L-function in (1) and the L-function in Theorem 1.1.1. From [Yam2014, Theorem 9.1], one sees that the L-function of π_W

$$L(s, \mathrm{BC}(\pi_W) \otimes \mu^n)$$

is holomorphic and nonzero at $s = \frac{1}{2}$. By Proposition 8.1.4,

$$L(s, \mathrm{BC}(\pi_V) \times \mathrm{BC}(\pi_W)) = L(s, \mathrm{BC}(\pi_{W_0}) \times \mathrm{BC}(\sigma_{W_0}) \otimes \mu^{-1}) L(s, \mathrm{BC}(\pi_W) \otimes \mu^n).$$

Therefore

$$L\left(\frac{1}{2}, \mathrm{BC}(\pi_V) \times \mathrm{BC}(\pi_W)\right) \neq 0$$

is equivalent to

$$L\left(\frac{1}{2}, \mathrm{BC}(\pi_{W_0}) \times \mathrm{BC}(\sigma_{W_0}) \otimes \mu^{-1}\right) \neq 0.$$

Let us prove (1) implies (2). Assume (1). We apply Theorem 1.1.1. There is a Hermitian space W_0', and irreducible automorphic representations $\pi_{W_0'}$ and $\sigma_{W_0'}$ of $U(W_0')(\mathbb{A}')$, which are nearly equivalent to π_{W_0} and σ_{W_0} respectively, such that the Fourier–Jacobi period

$$\int_{U(W_0')(k')\backslash U(W_0')(\mathbb{A}')} \overline{\varphi_{\pi_{W_0'}}(g)\varphi_{\sigma_{W_0'}}(g)}\theta_{W_0'}(g, 1, \phi_{W_0'})\mathrm{d}g$$

is not identically zero. Since $L(\frac{1}{2}, \mathrm{BC}(\pi_W) \otimes \mu^n) \neq 0$, one sees that there is a Hermitian space W', such that $\pi_{W_0'}$ is the theta lift of some automorphic representation $\pi_{W'}$ of $U(W')(\mathbb{A}')$. Therefore we can take $\varphi_{\pi_{W_0'}}(g)$ to be of the form

$$\int_{U(W')(k')\backslash U(W')(\mathbb{A}')} \overline{\varphi_{\pi_{W'}}(h)\theta_{W',W_0'}(g, h, \phi_{W,W'})}\mathrm{d}h.$$

From the seesaw identity one gets

$$\int_{U(W_0')(k')\backslash U(W_0')(\mathbb{A}')} \varphi_{\sigma_{W_0'}}(h) \left(\int_{U(W')(k')\backslash U(W')(\mathbb{A}')} \varphi_{\pi_{W'}}(g)\theta_{W,W_0'}(g, h, \phi_n)\mathrm{d}g\right) \theta_{W_0'}(1, h, \phi_1)\mathrm{d}h$$

$$= \int_{U(W')(k')\backslash U(W')(\mathbb{A}')} \varphi_{\pi_{W'}}(g) \left(\int_{U(W_0')(k')\backslash U(W_0')(\mathbb{A}')} \overline{\varphi_{\sigma_{W_0'}}(h)}\theta_{V',W_0'}(g, h, \phi)\mathrm{d}h\right) \mathrm{d}g,$$

where $W' \subset V'$ is a relevant pair of $W \subset V$ and $\phi = \phi_n \otimes \phi_1$. The change of order of integration is justified since $\varphi_{W'}$ and $\varphi_{W_0'}$ are cusp forms and the theta series is of moderate growth.

This particularly shows that the theta lift of $\sigma_{W_0'}$ to $U(V')(\mathbb{A}')$ is nonzero. Denote this theta lift by $\pi_{V'}$. It follows from Proposition 8.1.4 that the representations $\pi_{V'}$ (resp. $\pi_{W'}$) and π_V (resp. π_W) are nearly equivalent. This proves (2).

Now let us prove that (2) implies (1). We use only the fact that π_V is the theta lift of σ_{W_0}. So without loss of generality, one may assume that $W' = W$ and $V' = V$. Then there are choices of $\varphi_W \in \pi_W$ and $\varphi_V \in \pi_V$ such that

$$\int_{U(W)(k')\backslash U(W)(\mathbb{A}')} \varphi_{\pi_W}(g)\varphi_{\pi_V}(g)\mathrm{d}g \neq 0.$$

Since π_V is the theta lift of σ_{W_0}, one can take

$$\varphi_{\pi_V} = \theta_{W_0,V}(\varphi_{\sigma_{W_0}}, \phi_{W_0,V}).$$

Moreover, one can take $\phi_{W_0,V} = \phi_n \otimes \phi_1$ with $\phi_n \in \mathcal{S}(\mathbf{L}_n(\mathbb{A}'))$ and $\phi_1 \in \mathcal{S}(\mathbf{L}(\mathbb{A}'))$. Then by the seesaw argument again, one has

$$\int_{U(W_0)(k')\backslash U(W_0)(\mathbb{A}')} \overline{\varphi_{\sigma_{W_0}}(h)} \left(\int_{U(W)(k')\backslash U(W)(\mathbb{A}')} \varphi_W(g)\theta_{W,W_0}(g,h,\phi_n)\mathrm{d}g \right) \theta_{W_0}(1,h,\phi_1)\mathrm{d}h \neq 0.$$

In particular, the inner integral does not vanish. This shows that the theta lift $\theta_{W,W_0}(\pi_W) \neq 0$. This theta lift is $\overline{\pi_{W_0}}$ by Proposition 8.1.1. Applying Theorem 1.1.1,

$$L\left(\frac{1}{2}, \mathrm{BC}(\pi_{W_0}) \times \mathrm{BC}(\sigma_{W_0}) \otimes \mu^{-1}\right) \neq 0.$$

This proves (1). $\qquad\square$

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Department of Mathematics, Columbia University, New York, NY, 10027

E-mail address: xuehang@math.columbia.edu