INTEGRALS AND SERIES

[7.7] Definition of convergence of improper integrals:

Suppose $f(x)$ is positive for $x \geq a$.

If $\lim_{b\to\infty}\int_a^b f(x)\,dx$ is a finite number, we say that $\int_a^\infty f(x)\,dx$ **converges** and define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx .$$

Otherwise, we say that the integral **diverges**.

[7.8] Comparison Test for $\int_a^\infty f(x)\,dx$

Assume $f(x)$ is positive. Proving convergence or divergence involves two stages:

(1) By looking at the behavior of the integrand for large *x*, guess whether the integral converges or not.

(2) Confirm the guess by finding an appropriate function and inequality so that:

If $0 \leq f(x) \leq g(x)$ and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

If $0 \leq g(x) \leq f(x)$ and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

[7.8] Useful Integrals for Comparison

(1) $\int_1^\infty \frac{1}{x^p}\,dx$ converges to $1/(p-1)$ for $p > 1$ and diverges for $p \leq 1$.

(2) $\int_0^1 \frac{1}{x^p}\,dx$ converges for $p < 1$ and diverges for $p \geq 1$.

(3) $\int_0^\infty e^{-ax}\,dx$ converges for $a > 0$.

[9.2] Infinite Geometric Series

If $|x| < 1$, $\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$

[9.3] Connection between Series and Integrals – The Integral Test

Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive for $x \geq c$.

If $\int_c^\infty f(x)\,dx$ converges, then $\sum a_n$ converges.

If $\int_c^\infty f(x)\,dx$ diverges, then $\sum a_n$ diverges.

[9.3] A Useful Series for Comparison

The p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

[9.4] Comparison Test

Suppose $0 \le a_n \le b_n$ for all n.

If $\sum b_n$ converges, then $\sum a_n$ converges.

If $\sum a_n$ diverges, then $\sum b_n$ diverges.

[9.4] Limit Comparison Test

Suppose $a_n > 0$ and $b_n > 0$ for all n.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $c > 0$, then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

[9.4] Convergence of Absolute Value

If $\sum |a_n|$ converges, then so does $\sum a_n$.

[9.4] The Ratio Test

For a series $\sum a_n$, suppose the sequence of ratios $\left|\frac{a_{n+1}}{a_n}\right|$ has a limit: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$,

If $L < 1$, then $\sum a_n$ converges.

If $L > 1$ or if L is infinite, then $\sum a_n$ diverges.

If $L = 1$, the test does not tell us anything about the convergence of $\sum a_n$.

[9.4] Alternating Series Test

The alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges if $0 < a_{n+1} < a_n$ for all n and $\lim_{n\to\infty} a_n = 0$.

[9.5] Power Series – Radius of Convergence (ROC or R) and Interval of Convergence (IOC)

For the power series $\sum_{n=0}^{\infty} C_n (x-a)^n$:

- If $\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right|$ is infinite, then $R = 0$ and the series converges only for $x = a$.
- If $\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = 0$, then $R = \infty$ and the series converges for all values of x.
- If $\lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| = K$, where K is finite and nonzero, then $R = 1/K$ and the series converges for $|x-a| < R$ and diverges for $|x-a| > R$.

Courtesy of Faith Bridges