INFINITE-DIMENSIONAL ANALYSIS AND QUANTUM THEORY—A TOPICS COURSE PROPOSAL

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Many mathematical concepts of central importance grew out of theoretical physics. In the past century, quantum theory was a particularly fruitful source of inspiration for several fields of mathematics, including analysis and probability theory. The mathematical language of quantum physics is the theory of operators in Hilbert spaces. The need to consider quantum fields and systems with variable number of particles led to the concept of the Fock space. This beautiful mathematical structure formalizes the second quantization approach developed by the physicists. Characteristically, this is where physics and mathematics meet—Vladimir Fock was a physicist and Fock space is a mathematical object. Theory of operators in the Fock space furnishes a mathematical description of key elements of modern quantum theory: multiparticle Hamiltonians, creation and annihilation operators, the Weyl group, coherent states—to mention a few. It was for his work on coherent states and their key role in quantum optics that Roy Glauber was awarded the 2005 Nobel prize in physics. It was not immediately realized—and it took Norbert Wiener to explicitly prove it—that bosonic Fock spaces can be represented as spaces of random variables. This fruitful idea was exploited in quantum theory by mathematical physicists, in particular by Irving Segal. On the other hand, a probabilistic motivation and applications to partial differential equations led Paul Malliavin to develop analysis in the infinite-dimensional space of continuous functions, equipped with the Wiener measure. The key ingredients of Malliavin calculus can be interpreted in the language of second quantization (interestingly, Malliavin was apparently initially not aware of this). The second quantization and analysis in infinite-dimensional spaces of random variables are among the most successful theories in physics and mathematics respectively. Their correspondence bore a further fruit—quantum stochastic calculus, a key tool in the mathematical description of open quantum systems—systems exchanging energy with the environment. Quantum stochastic analysis—inspired by physics and developed mathematically by Robin Hudson and Kalyanapuram Parthasarathy is a beautiful example of interaction between the two fields. Probabilistic interpretation of the Hudson-Parthasarathy equations leads to the quantum trajectories approach in the theory of open quantum systems. Quantum trajectories played a crucial role in the interpretation of optical Schrödinger cat experiments which won Serge Haroche the 2012 Nobel prize in physics.

The course will be an introduction to these exciting developments, aimed at graduate students in mathematics, physics, optical sciences—and everybody else who is interested in the subject. I will only assume that the audience is familiar with the basics of Hilbert space theory and probability. The topics I am planning to discuss are:

- Bosonic Fock spaces, exponential and coherent vectors, basic operators
- Mathematics of second quantization, Weyl group
- Canonical commutation relations
- Fermionic Fock spaces
- Gaussian random variables and Gaussian families
- A short introduction to Itô stochastic calculus
- Wiener chaos representation of square-integrable functionals of the Wiener process and its analog for the Poisson process
- Introduction to Malliavin calculus—calculus on the Wiener and Poisson spaces

• Hörmander hypoellipticity theorem

• Introduction to quantum probability and quantum stochastic calculus. Quantum trajectories.

Course objective and learning goals: the main objective is to introduce the audience to the mathematical description of many-body quantum systems via the Fock space formalism, known as second quantization. Virtually the same mathematical structure arises in the analysis of functionals of Wiener and Poisson stochastic processes. The students will learn both points of view with important applications: representation of canonical commutation relations in quantum theory and Malliavin's proof of regularity of solutions to partial differential equations satisfying Hörmander condition. They will become thoroughly familiar with Fock spaces and operators on them, which are among the most important object in analysis and mathematical physics. They will also be introduced to the analysis on infinite-dimensional spaces—the Wiener and Poisson spaces. The course will provide and introduction to the quantum stochastic calculus—an important tool used by theoretical and mathematical physicists to model evolution of open quantum systems. Open problems and research opportunities will be discussed. The interdependence of mathematical and physical ideas will be stressed throughout the course.

Literature: I will not use a single text. The main sources will be:

1. Ph. A. Martin, F. Rothen: Many-Body Problems and Quantum Field Theory. Springer 2004

2. D. Nualart, E. Nualart: Introduction to Malliavin Calculus. Cambridge University Press 2018

3. K. R. Parthasarathy: An Introduction to Quantum Stochastic Calculus. Birkhaüser 1992

4. S. Haroche, J.-M. Raimond: Exploring the quantum: atoms, cavities and photons. Oxford University Press 2006