# Algebra Qualifying Examination

## August 2006

Do either one of $nA$ or $nB$ for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say.

1A A finite group is said to be a quasi-$p$ group if it is generated by its $p$-sylow subgroups. Give an example of a non-solvable group $G$ which is a quasi 2, quasi-3 and a quasi-5 group simultaneously.

1B Suppose that $G$ is a group of order $p^2q$ where $p, q$ are prime numbers. Show that $G$ is solvable.

2A The ideal $(7, X^3 + 2X + 1) \subset \mathbb{Z}[X]$ is a prime ideal. Prove or disprove.

2B Show that the ideal $(3, X^3 - X^2 + 2X - 1)$ in $\mathbb{Z}[X]$ is not principal.

3A Show that a $2 \times 2$-matrix $M$ with entries in $\mathbb{Q}$ and with positive determinant can be written as

$$M = SB$$

where $S$ is a $2 \times 2$-matrix with integer entries and determinant one and $B$ is an upper triangular matrix with entries in $\mathbb{Q}$.

3B Let $G = SL_3(\mathbb{F}_7)$ be the group of $3 \times 3$ matrices with determinant one and entries in $\mathbb{F}_7$ (here $\mathbb{F}_7$ is the field with seven elements). Show that the set of upper triangular matrices with ones on the diagonal is a subgroup of $G$ and that it is a 7-Sylow subgroup of $G$.

4A An element $r$ of a ring $R$ is said to be nilpotent if $r^n = 0$ for some $n \geq 1$. Now let $R$ be a commutative ring and let $R[X]$ be the ring of polynomials in $X$ with coefficients in $R$. The a polynomial

$$a_0 + a_1X + \ldots + a_nX^n \in R[X]$$

is nilpotent in $R[X]$ if and only if $a_0, a_1, \ldots, a_n$ are nilpotent in $R$.

4B Suppose $A = \mathbb{C}[t^2, t^3] \subset \mathbb{C}[t]$ (here $A$ is the subset consisting of all complex coefficient polynomials in $t^2, t^3$. For instance $t^4 + 2t^6$ is in $A$ but $t^2 + t + 1$ is not). Show that $A$ is a Noetherian ring.

5A Let $K$ be the field obtained by attaching to $\mathbb{Q}$ all the roots of $X^p - 2 = 0$ where $p \geq 3$ is a prime number. Calculate the Galois group of $K/\mathbb{Q}$.

5B Let $\zeta = \cos(\pi/6) + i\sin(\pi/6) \in \mathbb{C}$. Calculate the degree of $\mathbb{Q}(\zeta)/\mathbb{Q}$ and find the minimal polynomial of $\zeta$ over $\mathbb{Q}$.