ALGEBRA QUALIFYING EXAMINATION

AUGUST 2007

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say. Any ring denoted R is a commutative ring with identity.

1A. Let K be a field. Suppose
(i) n is an integer not divisible by the characteristic of K, and
(ii) K contains all the nth roots of 1 that lie in its algebraic closure.
Show that any $n \times n$ matrix with entries in K whose characteristic polynomial is $\lambda^n - 1$ is diagonalizable over K. Give examples to show that both hypotheses (i) and (ii) are neccessary.

1B. The $(n-1)$-dimensional real projective space $\mathbb{P}^{n-1}(\mathbb{R})$ is the quotient of the set $\mathbb{R}^n \setminus \{0\}$ by the equivalence relation $v \sim \lambda v$ for all $\lambda \in \mathbb{R}^{\times}$. Let A be an invertible $n \times n$ real matrix. Explain how the map $v \mapsto Av$ defines a map $\overline{A} : \mathbb{P}^{n-1}(\mathbb{R}) \to \mathbb{P}^{n-1}(\mathbb{R})$. In which of the following cases must there definitely exist $x \in \mathbb{P}^{n-1}(\mathbb{R})$ such that $\overline{A}(x) = x$? (Justify your answers.)
(a) A is upper triangular.
(b) A is symmetric.
(c) $A^{\mathrm{T}}A = I$.

2A. List the groups of order 6 and 8, up to isomorphism. (You do not need to prove that your list is complete.) Which of these are isomorphic to a subgroup of S_4? Of S_8?

2B. Show that for $n \geq 3$, the symmetric group S_n has trivial center.

3A. Let R be a principal ideal domain. Prove that every finitely generated projective module over R is free.

3B. Suppose that M is a finitely generated free module over a ring R. Determine the center of the endomorphism ring $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$.

4A. Let R be a noetherian integral domain. Show that R is a factorization domain, i.e., every non-zero element is a product of irreducible elements.

4B. Determine the kernel of the $\mathbb{C}$-algebra map $f : \mathbb{C}[x, y] \to \mathbb{C}[t]$ with $f(x) = t^2$ and $f(y) = t^3$.

5A. Let F be a field of characteristic not equal to two or three. Let $f(X) = X^3 + aX + b$ be an irreducible polynomial with coefficients in F. Let E be the splitting field of $f(X)$. Prove that the Galois group of $f(X)$ is S_3 if and only if the discriminant $\Delta = (\alpha_1 - \alpha_2)^2(\alpha_2 - \alpha_3)^2(\alpha_1 - \alpha_3)^2$ is not a perfect square in F.

5B. Prove that $\mathbb{C}(t + t^{-1})$ is the fixed field of $\mathbb{C}(t)$ under the automorphism $f(t) \mapsto f(1/t)$.