ALGEBRA QUALIFYING EXAMINATION

AUGUST 2008

Do either one of \( nA \) or \( nB \) for \( 1 \leq n \leq 5 \). Justify all your answers. Say what you mean, mean what you say. Any ring denoted \( R \) is a commutative ring with identity.

1A. Let \( A \in M_n(\mathbb{R}) \) be a positive definite symmetric matrix. Prove that there exists a unique positive definite symmetric \( B \in M_n(\mathbb{R}) \) such that \( B^2 = A \).

1B. State and prove the Vandermonde determinant formula.

2A. Let \( Q \) be the quaternion group \( \{\pm 1, \pm i, \pm j, \pm k\} \) of order 8. Prove that there is a homomorphism \( \text{Aut}(Q) \to S_3 \) whose kernel is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

2B. Let \( n > 2 \) be an integer and let \( H \) be a proper subgroup of \( A_n \). Prove that \( [A_n : H] \geq n \). Can this inequality be improved for any value of \( n \)?

3A. Prove or disprove: if \( R \) is a unique factorization domain, then the GCD of two elements \( a, b \in R \) is always expressible as \( \lambda a + \mu b \) for some \( \lambda, \mu \in R \).

3B. Let \( R = \mathbb{C}[0,1] \) be the ring of continuous complex-valued functions on the interval \([0,1]\). Let \( I_{\frac{1}{2}} \) be the elements of \( R \) with the property that \( f(\frac{1}{2}) = 0 \). Show that \( I_{\frac{1}{2}} \) is a maximal ideal of \( R \). Can you identify the quotient \( R/I_{\frac{1}{2}} \) with a field you know? Explain why \( R \) has an uncountable number of maximal ideals.

4A. Find (with proof) the smallest positive integer \( n \) such that an \( n^\circ \) angle is constructible by straightedge and compass.

4B. Let \( K = \mathbb{C}(t) \) be the field of rational functions in one variable \( t \) over the field \( \mathbb{C} \) of complex numbers. Let \( \omega \) be a non-trivial cube root of unity. Let \( \sigma, \tau \) be automorphisms of \( K \) fixing \( \mathbb{C} \) such that \( \sigma(t) = \omega t \) and \( \tau(t) = t^{-1} \). Show that the subgroup \( \langle \sigma, \tau \rangle \subset \text{Aut}(K) \) is isomorphic to \( S_3 \), and that the fixed field \( K^{\langle \sigma, \tau \rangle} \) is equal to \( \mathbb{C}(t^3 + t^{-3}) \).

5A. Let \( V \) be a vector space over a field \( K \), and let \( e, f \in V \) be two linearly independent vectors. Prove that the element \( e \otimes f + f \otimes e \in V \otimes_K V \) is not equal to any basic tensor of the form \( v \otimes w \) with \( v, w \in V \).

5B. An abelian group has generators \( a, b, c, d \) and defining relations \( 2a - 2c = 0 \), \( 4b - 8d = 0 \), \( 6a + 4b + c - d = 0 \), \( 2a + 4b + 5c - d = 0 \). Express the group as a direct sum of cyclic groups.