ALGEBRA QUALIFYING EXAMINATION

AUGUST 2008

Do either one of nA or nB for $1 \le n \le 5$. Justify all your answers. Say what you mean, mean what you say. Any ring denoted R is a commutative ring with identity.

1A. Let $A \in M_n(\mathbb{R})$ be a positive definite symmetric matrix. Prove that there exists a unique positive definite symmetric $B \in M_n(\mathbb{R})$ such that $B^2 = A$.

1B. State and prove the Vandermonde determinant formula.

2A. Let Q be the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ of order 8. Prove that there is a homomorphism $\mathrm{Aut}(Q) \to S_3$ whose kernel is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

2B. Let $n > 2$ be an integer and let H be a proper subgroup of A_n. Prove that $[A_n : H] \ge n$. Can this inequality be improved for any value of n?

3A. Prove or disprove: if R is a unique factorization domain, then the GCD of two elements $a, b \in R$ is always expressible as $\lambda a + \mu b$ for some $\lambda, \mu \in R$.

3B. Let $R = \mathbb{C}[0,1]$ be the ring of continuous complex-valued functions on the interval $[0,1]$. Let $I_{\frac{1}{2}}$ be the elements of R with the property that $f(\frac{1}{2}) = 0$. Show that $I_{\frac{1}{2}}$ is a maximal ideal of R. Can you identify the quotient $R/I_{\frac{1}{2}}$ with a field you know? Explain why R has an uncountable number of maximal ideals.

4A. Find (with proof) the smallest positive integer n such that an n° angle is constructible by straightedge and compass.

4B. Let $K = \mathbb{C}(t)$ be the field of rational functions in one variable t over the field $\mathbb{C}$ of complex numbers. Let ω be a non-trivial cube root of unity. Let σ, τ be automorphisms of K fixing $\mathbb{C}$ such that $\sigma(t) = \omega t$ and $\tau(t) = t^{-1}$. Show that the subgroup $\langle \sigma, \tau \rangle \subset \mathrm{Aut}(K)$ is isomorphic to S_3, and that the fixed field $K^{\langle \sigma, \tau \rangle}$ is equal to $\mathbb{C}(t^3 + t^{-3})$.

5A. Let V be a vector space over a field K, and let $e, f \in V$ be two linearly independent vectors. Prove that the element $e \otimes f + f \otimes e \in V \otimes_K V$ is not equal to any basic tensor of the form $v \otimes w$ with $v, w \in V$.

5B. An abelian group has generators a, b, c, d and defining relations $2a - 2c = 0$, $4b - 8d = 0$, $6a + 4b + c - d = 0$, $2a + 4b + 5c - d = 0$. Express the group as a direct sum of cyclic groups.