ALGEBRA QUALIFYING EXAM  
FALL 2009

- Do any one of the problems nA or nB where n = 1, 2, 3, 4, 5.
- You may use a separate sheet for scratch work.
- Be precise, concise and to the point.

1A: Let $A$ be an $n \times n$ matrix with complex entries. Assume that $A$ is nilpotent (i.e. $A^m = 0$ for some $m \geq 1$). Show that the trace of $A$ is zero.

1B: Let $A$ be the following $n$ by $n$ integer matrix for $n \geq 3$:

$$
A := \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & -1 & 2 & -1 \\
0 & \ldots & 0 & -1 & 2
\end{pmatrix}
$$

Compute the determinant of $A$.

2A: Let $G$ be a finite simple group of order 168. Show that there exists an injective homomorphism $G \hookrightarrow S_8$. Is there an injective homomorphism $G \hookrightarrow S_6$?

2B: Let $G$ be a finite group, $p$ a prime, $N$ a normal subgroup of $G$ and $P$ be a Sylow $p$-subgroup of $N$. Show that $G = N_G(P) \cdot N$.

3A: Let $R = \mathbb{C}[x]$. Determine all simple modules over $R$ up to isomorphism.

3B: Let $R$ be the subring of $\mathbb{Q}$ consisting of all $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b$ is odd. Show that $R$ is a principal ideal domain and determine all ideals of $R$.

4A: Show that for every $n \geq 1$, there exists an irreducible polynomial $f_n(X) \in \mathbb{Q}[x]$, of degree $n$. Show that this implies that $\overline{\mathbb{Q}}/\mathbb{Q}$ is not finite.

4B: Let $\alpha := \sqrt{2} + \sqrt{2} \in \mathbb{C}$. Determine the splitting field $K \subseteq \mathbb{C}$ of the minimal polynomial of $\alpha$ (over $\mathbb{Q}$); determine the Galois group of $K/\mathbb{Q}$ and all subfields of $K$.

5A: Find all semi-simple rings of order 1200.

5B: Let $G$ be an abelian group with generators $x, y, z, t$ and with defining relations $xy = z$, $yz = t$, $zt = x$, and $tx = y$. Write $G$ as a direct product of cyclic groups and determine whether there is a group homomorphism of $G$ onto $\mathbb{Z}$. 

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