ALGEBRA QUALIFYING EXAM
FALL 2012

- Do any one of the problems nA or nB where $n = 1, 2, 3, 4, 5$.
- You may use a separate sheet for scratch work.
- Be precise, concise and to the point.
- Each problem is worth 25 points.
- Show all steps and details; say what you mean, mean what you say.

1A) Determine the number of conjugacy classes of elements of order 2 in $\mathrm{GL}_n(F)$, where F is a field and $n \in \mathbb{N}, n > 0$.

1B) Let A and B be $n \times n$ complex matrices which satisfy $AB - BA = A$.
  (1) Prove that $A^k B - BA^k = kA^k$ for all positive integers k.
  (2) Prove that A is nilpotent. (Hint: fix B and consider the linear operator $T_B : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by $T_B(X) = XB - BX$).

2A) Let G be a group of order 55 and let Ω be a G-set of size 23. Show that there is an element $\omega \in \Omega$ such that for all $g \in G$ the following statement holds: $g\omega = \omega$.

2B) Let p and q be distinct primes and G a group of order pq.
  (1) Prove that G is solvable.
  (2) Prove that G is nilpotent if and only if G is cyclic.
  (3) Find primes p and q for which there is a non-nilpotent group of order pq, and justify your answer.

3A) Prove or disprove: (2i-1,2i+3) is a prime ideal in $\mathbb{Z}[i]$.

3B) Let K be a field and R a subring of K such that for all $x \in K^\times$ one has $x \notin R \implies x^{-1} \in R$.
  (1) Suppose $x \in K$ satisfies $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ with $a_i \in R$ for all i. Prove that $x \in R$.
  (2) Prove that any finitely generated ideal of R is principal.

4A) Let F be the field with 81 elements. Give all the subfields, determine how many elements α in F satisfy $F = \mathbb{F}_3(\alpha)$ and also determine the number of elements in F that generate F^* as a group.

4B) Let K be the splitting field of $x^5 - 6x + 3$ over the rationals. Determine (with proof) $\mathrm{Gal}(K/\mathbb{Q})$.

5A) Let R be a finite semisimple (nonzero) ring such that its size is not divisible by k^4 for any $k \in \mathbb{N}, k \geq 2$. Show that R is a direct product of fields.

5B) For integers a, b, c, let $G_{a,b,c}$ be the *abelian* group with presentation

$$G_{a,b,c} := \langle x, y, z \mid 6x + 6y + 4z = 0, \quad 4x + 10y + 4z = 0, \quad ax + by + cz = 0 \rangle.$$

  (1) What conditions on a, b, c guarantee that $G_{a,b,c}$ is finite?
  (2) Is there a triple of integers a, b, c for which $G_{a,b,c} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$? Justify your answer.
  (3) Is there a triple of integers a, b, c for which $G_{a,b,c} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$? Justify your answer.