ALGEBRA QUALIFYING EXAMINATION

JANUARY 2007

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say. Any ring denoted R is a commutative ring with identity.

1A. Let $A, B \in M_2(\mathbb{C})$ be 2×2 matrices with complex entries. If A is invertible and there is a positive integer n such that $ABA^{-1} = B^n$, prove that either B is diagonalizable, or else the eigenvalues of A have ratio n.

1B. Suppose that A is an $n \times n$ matrix with complex entries such that for every $n \times n$ matrix B with complex entries we have $\mathrm{Trace}(AB) = 0$. Show that $A = 0$.

2A. Let p, q, r be distinct primes. Prove that a group of order pqr cannot be a simple group.

2B. Let p be a prime, and suppose G is a group of order p^n. Let H be the intersection of all subgroups of G of order p^{n-1}. Prove that H is a normal subgroup of G, and that the quotient G/H is abelian.

3A. Let K be a field. Determine all polynomials $p(x) \in K[x]$ such that the ring $K(x)[y]/(x^2 + y^3 + y^2 - p(x)y)$ is *not* a field. (Recall that $K(x)$ denotes the field of rational functions in x.)

3B. Let R be a ring, and consider the ring $R[\![x]\!]$ of formal power series with coefficients in R. Prove that $\sum_{i=0}^{\infty} a_i x^i \in R[\![x]\!]^\times$ if and only if $a_0 \in R^\times$.

4A. Suppose $K/\mathbb{Q}$ is a field extension with $[K : \mathbb{Q}] = 4$. Must there exist a field L lying between K and $\mathbb{Q}$ with $[L : \mathbb{Q}] = 2$?

4B. Let K be the field obtained by attaching to $\mathbb{Q}$ all the roots of $X^p - 2 = 0$ where p is a prime number. Show that the Galois group of $K/\mathbb{Q}$ is a solvable group.

5A. Suppose that $E, F \subset \mathbb{C}$ are fields which are finite extensions of $\mathbb{Q}$. Assume further that $E \cap F = \mathbb{Q}$. Prove that the $\mathbb{Q}$-algebra $E \otimes_{\mathbb{Q}} F$ is actually a field.

5B. Suppose that R is a ring such that every submodule of a finitely generated free R-module is free. Show that R is a PID.