ALGEBRA QUALIFYING EXAMINATION

JANUARY 2008

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say. Any ring denoted R is a commutative ring with identity.

1A. Let A be an element in $\mathrm{GL}(2, \mathbb{Q})$, the group of 2×2 invertible matrices with entries in $\mathbb{Q}$. Suppose that A has finite order. Find all possibilities for the order of A, and give an example illustrating each possible order.

1B. Calculate the determinant of the n-by-n matrix

$$\begin{pmatrix} x & -1 & 0 & \cdots & 0 & 0 \\ 0 & x & -1 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & x & -1 \\ x-1 & x-1 & x-1 & \cdots & x-1 & x-1 \end{pmatrix}$$

whose jth row (for $j < n$) has x and -1 in columns j and $j+1$ respectively (and zeros elsewhere), and whose nth row has all entries equal to $x-1$.

2A. Let G be a group and let $f : G \to G$ be an automorphism of G with the property that $f(g) = g$ if and only if $g = 1$. Suppose G is finite. Show that every element of G is of the form $f(g)g^{-1}$.

2B. List four non-isomorphic groups of order 63, and prove that no two of the groups you give are isomorphic to one another.

3A. (a) Give an example of a ring R and three prime ideals $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2$ in R such that $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2$.
(b) Give an example of a ring R that does *not* contain three prime ideals $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2$ such that $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2$.

3B. (a) Write the quotient $\mathbb{R}[x]/(x^4 - 1)$ as a product of fields.
(b) Prove that the quotient $\mathbb{R}[x]/(x^4 + x^2)$ cannot be written as a product of fields.

4A. Find the Galois group of $\mathbb{Q}(\cos(2\pi/11))$ over $\mathbb{Q}$.

4B. Find an irreducible polynomial of degree 3 with rational coefficients whose Galois group is not S_3.

5A. Prove or disprove: $\mathbb{Q}$ is a free $\mathbb{Z}$-module.

5B. Suppose R is a ring. Let M be an R-module. Show that M is Noetherian if and only if every R-submodule of M is finitely generated.