ALGEBRA QUALIFYING EXAMINATION

JANUARY 2009

Do either one of \( nA \) or \( nB \) for \( 1 \leq n \leq 5 \). Justify all your answers. Say what you mean, mean what you say. Any ring denoted \( R \) is a commutative ring with identity.

1A. Let \( M \) be a \( 3 \times 3 \) matrix with entries in \( \mathbb{C} \) and suppose that for every \( 3 \times 3 \) matrix \( A \) with complex entries, we have \( \text{Trace}(MA) = 0 \). Then show that \( M = 0 \).

1B. Let \( m \) be a positive integer and suppose that \( c_1, \ldots, c_n \in \mathbb{Q} \) have the property that \( \sum_{k=1}^{n} c_k k^j = m^j \) for each \( j = 0, \ldots, n-1 \). Use Cramer’s rule to compute \( c_n \).

2A. Let \( G \) be a finite simple group. Let \( p \) be a prime dividing its order. Prove or disprove the following statement: \( G \) is generated by its \( p \)-Sylow subgroups.

2B. Prove that the additive groups \( \mathbb{Z}[1/2] \) and \( \mathbb{Z}[1/3] \), consisting of rational numbers whose denominators are powers of 2 and 3 respectively, are not isomorphic.

3A. (a) Give an example of a UFD that is not a PID.
(b) Let \( R \) be a UFD, and let \( P \) be any nonzero prime ideal of \( R \) such that there are no prime ideals lying strictly between \((0)\) and \( P \). Prove that \( P \) is principal.

3B. Suppose \( R \) is a commutative ring with the property that for every \( x \in R \), we have \( x^2 = x \). Show that (1) \( R \) has characteristic two, (2) every prime ideal \( P \) of \( R \) is maximal with quotient \( R/P \simeq \mathbb{Z}/2 \).

4A. Let \( f(x) = x^3 - 7 \in \mathbb{Q}[x] \). Show that \( f \) is an irreducible polynomial and compute its Galois group.

4B. Give an example (with proofs) of a field \( K/\mathbb{Q} \) such that \( \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4 \mathbb{Z} \times \mathbb{Z}/2 \mathbb{Z} \).

5A. Here is a list of five \( \mathbb{R} \)-algebras: \( \mathbb{R}^4 \), \( \mathbb{R}^2 \times \mathbb{C} \), \( \mathbb{C} \times \mathbb{C} \), \( \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \), \( \mathbb{R}[x]/(x^4-1) \). This list contains two pairs of isomorphic \( \mathbb{R} \)-algebras, and one “odd one out”. Determine (with proof) the two pairs of isomorphic \( \mathbb{R} \)-algebras.

5B. Prove or disprove: the map \( G \mapsto Z(G) \) which sends a group \( G \) to its center \( Z(G) \) can be made into an isomorphism preserving functor from the category of groups to itself.