ALGEBRA QUALIFYING EXAMINATION

JANUARY 2009

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say. Any ring denoted R is a commutative ring with identity.

1A. Let M be a 3×3 matrix with entries in $\mathbb{C}$ and suppose that for every 3×3 matrix A with complex entries, we have $Trace(MA) = 0$. Then show that $M = 0$.

1B. Let m be a positive integer and suppose that $c_1, \ldots, c_n \in \mathbb{Q}$ have the property that $\sum_{k=1}^{n} c_k k^j = m^j$ for each $j = 0, \ldots, n-1$. Use Cramer's rule to compute c_n.

2A. Let G be a finite simple group. Let p be a prime dividing its order. Prove or disprove the following statement: G is generated by its p-Sylow subgroups.

2B. Prove that the additive groups $\mathbb{Z}[1/2]$ and $\mathbb{Z}[1/3]$, consisting of rational numbers whose denominators are powers of 2 and 3 respectively, are not isomorphic.

3A. (a) Give an example of a UFD that is not a PID.
(b) Let R be a UFD, and let P be any nonzero prime ideal of R such that there are no prime ideals lying strictly between (0) and P. Prove that P is principal.

3B. Suppose R is a commutative ring with the property that for every $x \in R$, we have $x^2 = x$. Show that (1) R has characteristic two, (2) every prime ideal P of R is maximal with quotient $R/P \simeq \mathbb{Z}/2$.

4A. Let $f(x) = x^3 - 7 \in \mathbb{Q}[x]$. Show that f is an irreducible polynomial and compute its Galois group.

4B. Give an example (with proofs) of a field $K/\mathbb{Q}$ such that

$$\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

5A. Here is a list of five $\mathbb{R}$-algebras: $\mathbb{R}^4$, $\mathbb{R}^2 \times \mathbb{C}$, $\mathbb{C} \times \mathbb{C}$, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathbb{R}[x]/(x^4 - 1)$. This list contains two pairs of isomorphic $\mathbb{R}$-algebras, and one "odd one out". Determine (with proof) the two pairs of isomorphic $\mathbb{R}$-algebras.

5B. Prove or disprove: the map $G \mapsto Z(G)$ which sends a group G to its center $Z(G)$ can be made into an isomorphism preserving functor from the category of groups to itself.