ALGEBRA QUALIFYING EXAM
SPRING 2011

- Do any one of the problems nA or nB where $n = 1, 2, 3, 4, 5$.
- You may use a separate sheet for scratch work.
- Be precise, concise and to the point.
- Each problem is worth 25 points.
- Show all steps and details; say what you mean, mean what you say.

1A: Prove that an $n \times n$ matrix A with entries in $\mathbb{C}$ satisfying $A^3 = A$ can be diagonalized. Is the same true over any field? Prove or give a counterexample.

1B: Let F be a field, V a finite dimensional F-vector space, and let S, T be linear transformations from V to V such that $S \circ T = T \circ S$. Prove that if there is a basis B consisting of eigenvectors of S and a basis C consisting of eigenvectors of T, then there is a basis D consisting of eigenvectors of S and T.

2A: Prove that a group with 108 elements is not simple.

2B: Let G be a group generated by two elements a, b, where $a^2 = b^2 = 1$. Show that the commutator subgroup G' is cyclic.

3A: Let k be a field and let $a_1, \ldots, a_n$ be elements in k. Let $R = k[X_1, \ldots, X_n]$ be the polynomial ring in n variables. Prove that the ideal generated by

$$X_1 - a_1, \ldots, X_n - a_n$$

is a maximal ideal in R. (Hint: reduce to the case when $a_1 = \ldots = a_n = 0$ by considering a translation $X_1 \mapsto X_1 + a_1, \ldots X_n \mapsto X_n + a_n$.)

3B: Let R be a Principal Ideal Domain. Let $\mathfrak{p}$ be a prime ideal in $R[X]$ with the property that $\mathfrak{p} \cap R \neq \{0\}$. Prove that there is an irreducible element $\pi \in \mathfrak{p} \cap R$ such that either $\mathfrak{p} =< \pi >$ or $\mathfrak{p} =< \pi, f(X) >$ for some polynomial $f(X) \in R[X]$ which is irreducible mod π, i.e., in $S[X]$, where $S = R/ < \pi >$. (Hint: consider the quotient map $R[X] \to S[X]$.)

4A: Let K be a field and let $K(X)$ be the rational function field in the indeterminate X. Show that for any $T \in K(X)$ with $T \notin K$, the field $K(X)$ is algebraic over $K(T)$.

4B: Determine the splitting field, the subfields of the splitting field, and the Galois group of the polynomial $f = X^3 - 10$ over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-3})$.

5A: Let R be a commutative ring with $1 \neq 0$. Prove that if there is an isomorphism of R-modules $R^n \cong R^m$ then $n = m$.

5B: Suppose A and B are finite abelian groups each having all Sylow subgroups being cyclic; view A and B as $\mathbb{Z}$-modules. Calculate $A \otimes_{\mathbb{Z}} B$ and determine its Sylow subgroups.