ALGEBRA QUALIFYING EXAMINATION

JANUARY 2012

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say.

1A. Prove or disprove: every $n \times n$ real symmetric matrix is similar to a diagonal matrix.

1B. Let A and B be complex $n \times n$ matrices. Suppose that $A^3 = B^3 = 0$ and suppose that the rank of A is equal to the rank of B.
   a) For $n = 3$ show that A and B are similar.
   b) Is this still true when $n \geq 4$? Justify your answer!

2A. Let p be a prime and let $\mathbb{F}_p$ be the field with p elements. Prove that every Sylow p-subgroup of $\mathrm{GL}_3(\mathbb{F}_p)$ is conjugate to the subgroup $U_3 \subset \mathrm{GL}_3(\mathbb{F}_p)$ consisting of the upper triangular matrices with 1's on the diagonal.

2B. Describe two nonisomorphic, nonabelian groups of order 12 and show that they are nonisomorphic. For each group compute the derived subgroup and the center.

3A. Find all semisimple rings of order 240. State the results used in your proof.

3B. Let R be an integral domain with 1. Suppose R has finitely many ideals. Show that R is a field.

4A. Let $\alpha_n = \cos(\pi/2^n)$. For $n \geq 2$ show that $\mathbb{Q}(\alpha_n) \supset \mathbb{Q}(\alpha_{n-1})$ and find $[\mathbb{Q}(\alpha_n) : \mathbb{Q}]$.

4B. Let ϵ be a complex, primitive 15-th root of unity. Show that $K = \mathbb{Q}(\epsilon)$ is a Galois extension over $\mathbb{Q}$ and determine all its subfields. Prove or disprove: if α is a complex, primitive 41-th root of unity then $\mathbb{Q}(\alpha)$ is a subfield of K.

5A. Exhibit, with justification, the complete (up to isomorphism) list of all finitely generated $\mathbb{Z}$-modules M that satisfy the following condition: If $M \cong A \oplus B$ with $\mathbb{Z}$-modules A and B, then either $A = \{0\}$ or $B = \{0\}$.

5B. Let A be the abelian group given by the following presentation: $\langle x, y, z | 6x + 8y - 4z = 0,\ 8x + 4y + 6z = 0 \rangle$. Write A as a direct product of cyclic groups (up to isomorphism). Also, determine the number of homomorphisms from the symmetric group S_4 to A.