ALGEBRA QUALIFYING EXAMINATION
JANUARY 2013

Do either one of $nA$ or $nB$ for $1 \leq n \leq 5$. Justify all your answers. Say what you mean, mean what you say.

1A. Let $A \in M_n(F)$ be an $n \times n$ matrix with entries in an algebraically closed field $F$, and let $V \subseteq M_n(F)$ be the subset of matrices that commute with $A$. Prove that $V$ is an $F$-vector space of dimension $\geq n$.

1B. Let $F$ be the field with two elements, and let $M = (m_{i,j})_{1 \leq i,j \leq n}$ be the $n \times n$ matrix with $m_{i,j} = 1_F$ for all $i, j = 1, \ldots, n$. Determine the Jordan canonical form of $J$.

2A. Let $p > 0$ be a prime and $G$ a finite, simple group of order that is divisible by $p^2$. Prove that every proper subgroup of $G$ has index at least $2p$.

2B. Let $G$ be a group. Show that if Aut($G$) is a cyclic group, then $G$ is abelian. Hint: Consider the inner automorphism group.

3A. Let $R := M_2(Q)$ be the ring of $2 \times 2$ matrices with entries in $Q$.
   i) Exhibit a nonzero, proper left ideal of $R$.
   ii) Prove that $R$ is simple, i.e. that $R$ has no nonzero, proper two-sided ideals.

3B. Let $R$ be a ring with unity.
   i) If $R$ is commutative, show that the set of nilpotent elements of $R$ is an ideal in $R$.
   ii) Prove or disprove: If $R$ is arbitrary, then the set of nilpotent elements is an ideal.

4A. Determine, with proof, the number of distinct roots of $x^{35} - 1$ in the field with 64 elements.

4B. Suppose $F, K$, and $L$ are fields with $F \subseteq K \subseteq L$ and $[L : F]$ is finite. Either prove (using one of the equivalent definitions of Galois) or disprove (by exhibiting a counterexample) each of the following three assertions:
   i) If $L$ is Galois over $F$, then $L$ is Galois over $K$.
   ii) If $L$ is Galois over $F$, then $K$ is Galois over $F$.
   iii) If $L$ is Galois over $K$ and $K$ is Galois over $F$, then $L$ is Galois over $F$.

5A. Let $F$ be a field and set $R := F[x]$. Viewing $R$ as a module over itself via left multiplication, let $M := R^3 = R \oplus R \oplus R$, and let $N$ be the $R$-submodule of $M$ generated by $(x^2, x^3, x^4)$ and $(1, x + 1, x^2)$. Express the quotient $M/N$ explicitly as a direct sum of cyclic $R$-modules.

5B. Let $m, n \geq 1$. Describe the ring $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. In particular, what is the cardinality of this ring?