ALGEBRA QUALIFYING EXAMINATION

JANUARY 2014

Do either one of nA or nB for $1 \le n \le 5$. Justify all your answers. Say what you mean, mean what you say.

1A. Let A be an $n \times n$ real skew-symmetric matrix.
   i) Prove that the nonzero eigenvalues of A are purely imaginary.
   ii) Prove that $\det(A + I_n) \ge 1$.

1B. Let F be a field, and let $M_n(F)$ denote the ring of n-by-n matrices in F. Let $A \in M_n(F)$ with minimal polynomial equal to its characteristic polynomial. Suppose that $B \in M_n(F)$ commutes with A. Show that $B = f(A)$ for some $f \in F[x]$.

2A. Prove that a finite simple group of even order is generated by elements of order 2.

2B. Show that all groups of order $5 \cdot 7 \cdot 73$ are cyclic.

3A. Let R be a nonzero (not necessarily commutative) ring (with unity). If R is semisimple and of finite cardinality, prove that R is a direct product of fields.

3B. Consider the ring $R = \mathbb{Z}[x]/(x^2)$.
   i) Show that every ideal of R can be generated by two or fewer elements.
   ii) Show that (x) is the only prime ideal of R that is not maximal.

4A. Let $p_1 < p_2 < \cdots < p_r$ be positive prime numbers for some $r \ge 1$, and let K be the field extension of $\mathbb{Q}$ obtained by adjoining $\sqrt{p_i}$ for $1 \le i \le r$.
   i) Prove that K is a Galois extension of $\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q})$ an elementary abelian 2-group.
   ii) If $E \subset K$ is a subfield of K of degree 2 over $\mathbb{Q}$, prove that $E = \mathbb{Q}(\sqrt{m})$ for some m that is the product of all elements in a subset of $\{p_1, p_2, \ldots, p_r\}$.

4B. Let p be a prime, and let F be a field of characteristic not equal to p. Suppose that F contains a nontrivial pth root of unity, and let $a \in F^\times$.
   i) Show that a is a pth power in $F^\times$ if and only if $x^p - a$ is reducible in $F[x]$.
   ii) Let E be the splitting field of $x^p - a$, and suppose that a is not a pth power in $F^\times$. Determine $\mathrm{Gal}(E/F)$ up to isomorphism.

5A. Let R be a ring with unity. Let M be a left R-module in which the chains of distinct left R-submodules are of finite and bounded length. Let $f\colon M \to M$ be a left R-module endomorphism. Show that there exist f-stable left R-submodules U and N of M with $M = N \oplus U$ such that f restricts to an isomorphism of U and f restricts to a nilpotent endomorphism of N (i.e., $f^k(N) = 0$ for k sufficiently large).

5B. Let $f \in \mathbb{Q}[x]$ be a polynomial of degree $n \ge 1$ with exactly r real roots in $\mathbb{C}$.
   i) Show that f factors in $\mathbb{R}[x]$ as a product of r linear and $(n-r)/2$ quadratic polynomials. (You may use the fundamental theorem of algebra.)
   ii) Let $K = \mathbb{Q}[x]/(f)$. Show that
$$\mathbb{R} \otimes_{\mathbb{Q}} K \cong \mathbb{R}^r \times \mathbb{C}^{(n-r)/2}.$$