2006

Real Analysis Qualifying Exam, August ~~2005~~

Throughout the exam, identify the hypothesis and conclusion of every nontrivial theorem that you use.

1. Suppose that f is a real function on the line that is twice differentiable. Fix a real number x. Let $h(y) = f(y) + f'(y)(x-y)$.

   a. Find $h'(y)$.

   b. Suppose $a < x$. Apply the mean value theorem to h on the closed interval $[a, x]$ to prove that there is a c with $a < c < x$ and

   $$f(x) = f(a) + f'(a)(x-a) + f''(c)(x-c)(x-a).$$

   Note: The last term on the right hand side is Cauchy's form of the remainder for the Taylor polynomial approximation of degree one.

   c. Let $g(t) = x - \sqrt{t}$. Find the derivative of the composite function $h \circ g$.

   d. Suppose $g(r) = a$, so $a = x - \sqrt{r}$. Then g maps the closed interval $[0, r]$ to the closed interval $[a, x]$. Apply the mean value theorem to the composition $h \circ g$ on the closed interval $[0, r]$ to get a similar result, but with Lagrange's form of the remainder.

   e. The function g is not differentiable at zero, so $h \circ g$ need not be differentiable at x. Is this a problem? Explain.

2. Let $k \mapsto q_k$ be a bijection from the strictly positive natural numbers to the rational numbers. Show that

   $$\mathbf{R} \setminus \bigcup_{k=1}^{\infty} \left(q_k - \frac{1}{k^2}, q_k + \frac{1}{k^2} \right) \neq \emptyset.$$

3. Suppose that (X, ρ) is a metric space. Given a compact subset $K \subset X$ and a closed subset $B \subset X$, prove that if K and B are disjoint, then the distance between K and B is strictly positive.

4. Let $f(x) = \frac{\ln(x)}{x}$. Find the values of $p \geq 1$ such that $f \in L^p([1, \infty], dx)$.

5. Suppose that $f, g \in L^1(\mathbf{R}^n, dx)$. Define the convolution of f and g to be the function (defined for *a.e.* x) by

$$(f * g)(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\, dy.$$

a. Show that $f * g$ is well-defined and $f * g \in L^1(\mathbf{R}^n, dx)$.

b. Suppose for this part that $n = 1$ and that f is absolutely continuous with $f' \in L^1$. Show that $f * g$ is absolutely continuous, and calculate its derivative.

6. Determine whether the following statements are true or false, and briefly explain your answer:

a. $L^1([0,1], dx) \subset L^2([0,1], dx)$

b. $C([0,1])$ is a closed subspace of $L^1([0,1], dx)$.

c. l^2 and $L^2([0,1], dx)$ are not isomorphic as Hilbert spaces, because $\{1, 2, ..\}$ is countable, and $[0,1]$ is uncountable.

d. The dual of a separable normed linear space is itself a separable normed linear space.

7. Prove or disprove the following claim: The subset of $C([0,1])$ consisting of absolutely continuous functions ϕ with $\phi(0) = 0$ and

$$\int_0^1 |\phi'(x)|^p\, dx \le 1$$

is precompact, in $C([0,1])$, provided $p > 1$.

i.e., has compact closure,