Real Analysis Qualifying Exam, August 2007

Please show all your work; in particular, explain clearly all steps (such as interchanging limits) by quoting known theorems, and, when appropriate, by verifying that their assumptions are satisfied.

1. Show that $f_n(x) = \sqrt{x^2 + 1/n}$ converges to $f(x) = |x|$ uniformly on $\mathbf{R}$.

2. Let X be a measure space with finite measure $\mu(X) < +\infty$. A sequence of measurable functions f_n is said to converge to zero in measure if for each $\epsilon > 0$ the measure $\mu(\{x \mid |f_n(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$.

   a. Show that if $|f_n| \wedge 1$ (the minimum of $|f_n|$ and 1) converges to zero in L^1, then it converges to zero in measure.

   b. Show that if f_n converges to zero in measure, then $|f_n| \wedge 1$ converges to zero in L^1.

3. Let $f(x) \geq 0$ be in L^1 on the line with respect to Lebesgue measure. Let $g(x) = \sum_{n=-\infty}^{+\infty} f(x+n)$. Show that if g is in L^1, then $f = 0$ almost everywhere.

4. Show that

$$\int_{-\infty}^{+\infty} |f(x)|\, dx \leq \sqrt{2\pi} \left(\int_{-\infty}^{+\infty} |f(x)|^2\, dx \right)^{\frac{1}{4}} \left(\int_{-\infty}^{+\infty} x^2 |f(x)|^2\, dx \right)^{\frac{1}{4}}.$$

   Hint: Let $|f(x)| = 1/\sqrt{1 + a^2x^2} \cdot \sqrt{1 + a^2x^2}|f(x)|$. To get the result as stated you may want (at the end) to make an optimal choice of $a > 0$.

5. The functions $1, x, x^2, x^3, \ldots$ are each in $L^2([0,1])$, where $[0,1]$ is the unit interval equipped with Lebesgue measure. Let $\phi_0, \phi_1, \phi_2, \phi_3, \ldots$ be the orthonormal family generated from these by the Gram-Schmidt process. Explain why this is a maximal orthonormal family.

6. a. Let $X \subset \bar{X}$ be metric spaces with metrics that coincide on X. Suppose that X is a dense subspace of $\bar{X}$. Let Y be a complete metric space. Let $f : X \to Y$ be a uniformly continuous map of metric spaces. Show that there is a uniformly continuous map $\bar{f} : \bar{X} \to Y$ that extends f. (Be sure to prove that the map is a function: one input gives only one output.)

   b. Give an example to show that there is no such result on extension by continuity for the case when f is merely known to be continuous.

7. a. Consider a step function on $\mathbf{R}$ that is the indicator function of an interval $[a, b]$. This is clearly an element of $L^1(\mathbf{R})$ (where the real line $\mathbf{R}$ is equipped with Lebesgue measure), and it has $L^1(\mathbf{R})$ norm $b - a$. Calculate its Fourier transform and show that it is in $C_0(\mathbf{R})$, the space of functions that are continuous and vanish at infinity. Calculate the supremum norm of the Fourier transform.

   b. Use the first part to give a proof that the Fourier transform of every function in $L^1(\mathbf{R})$ is in $C_0(\mathbf{R})$.