Please show all your work; in particular, explain clearly all steps (such as interchanging limits) by quoting known theorems, and, when appropriate, by verifying that their assumptions are satisfied.

1. Show that \( f_n(x) = \sqrt{x^2 + 1/n} \) converges to \( f(x) = |x| \) uniformly on \( \mathbb{R} \).

2. Let \( X \) be a measure space with finite measure \( \mu(X) < \infty \). A sequence of measurable functions \( f_n \) is said to converge to zero in measure if for each \( \varepsilon > 0 \) the measure \( \mu(\{x \mid |f_n(x)| \geq \varepsilon\}) \to 0 \) as \( n \to \infty \).
   a. Show that if \( |f_n| \land 1 \) (the minimum of \( |f_n| \) and 1) converges to zero in \( \mathbb{L}^1 \), then it converges to zero in measure.
   b. Show that if \( f_n \) converges to zero in measure, then \( |f_n| \land 1 \) converges to zero in \( \mathbb{L}^1 \).

3. Let \( f(x) > 0 \) be in \( \mathbb{L}^1 \) on the line with respect to Lebesgue measure. Let \( g(x) = \sum_{n=-\infty}^{+\infty} f(x+n) \). Show that if \( g \) is in \( \mathbb{L}^1 \), then \( f = 0 \) almost everywhere.

4. Show that
   \[
   \int_{-\infty}^{+\infty} |f(x)| \, dx \leq \sqrt{2\pi} \left( \int_{-\infty}^{+\infty} |f(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{-\infty}^{+\infty} x^2 |f(x)|^2 \, dx \right)^{\frac{1}{2}}.
   \]
   Hint: Let \( |f(x)| = 1/\sqrt{1 + a^2 x^2} \cdot \sqrt{1 + a^2 x^2} |f(x)| \). To get the result as stated you may want (at the end) to make an optimal choice of \( a > 0 \).

5. The functions \( 1, x, x^2, x^3, \ldots \) are each in \( \mathbb{L}^2([0, 1]) \), where \([0, 1]\) is the unit interval equipped with Lebesgue measure. Let \( \phi_0, \phi_1, \phi_2, \phi_3, \ldots \) be the orthonormal family generated from these by the Gram-Schmidt process. Explain why this is a maximal orthonormal family.

6. a. Let \( X \subseteq \tilde{X} \) be metric spaces with metrics that coincide on \( X \). Suppose that \( X \) is a dense subspace of \( \tilde{X} \). Let \( Y \) be a complete metric space. Let \( f : X \to Y \) be a uniformly continuous map of metric spaces. Show that there is a uniformly continuous map \( \tilde{f} : \tilde{X} \to Y \) that extends \( f \). (Be sure to prove that the map is a function: one input gives only one output.)
   b. Give an example to show that there is no such result on extension by continuity for the case when \( f \) is merely known to be continuous.

7. a. Consider a step function on \( \mathbb{R} \) that is the indicator function of an interval \([a, b]\). This is clearly an element of \( \mathbb{L}^1(\mathbb{R}) \) (where the real line \( \mathbb{R} \) is equipped with Lebesgue measure), and it has \( \mathbb{L}^1(\mathbb{R}) \) norm \( b - a \). Calculate its Fourier transform and show that it is in \( \mathbb{C}_0(\mathbb{R}) \), the space of functions that are continuous and vanish at infinity. Calculate the supremum norm of the Fourier transform.
   b. Use the first part to give a proof that the Fourier transform of every function in \( \mathbb{L}^1(\mathbb{R}) \) is in \( \mathbb{C}_0(\mathbb{R}) \).