ANALYSIS QUALIFYING EXAM
AUGUST, 2008

PROBLEM 1

Find all pairs of positive numbers α and β for which the function $x^{\alpha}\cos(x^{-\beta})$ is of bounded variation on the interval $[0,1]$.

PROBLEM 2

Find the limit

$$\lim_{n\to\infty} n\int_0^{\infty} \frac{\arctan x}{1+n^2x^2}.$$

Justify all steps.

PROBLEM 3

Let $F:[0,1]^2\to[0,1]$ be a mapping given by the formula $F(x,y)=xy$. Let $\mathfrak{B}_{[0,1]}$ be the Borel sigma algebra on $[0,1]$, and let $\mathfrak{M}_F=\{F^{-1}(B): B\in\mathfrak{B}_{[0,1]}\}$. Show that $\mathfrak{M}_F$ is a sigma algebra of subsets of $[0,1]^2$. Describe functions $f(x,y)$ that are $\mathfrak{M}_F$-measurable.

PROBLEM 4

Let $f(x)\in L^1([0,1])$. Suppose that

$$\int_a^b f(x)dx=(b-a)\int_0^1 f(x)dx$$

for all rational numbers a and b. What can be said about the function $f(x)$?

PROBLEM 5

Let $f(x)\in L^2((-\infty,\infty))$. Suppose that

$$\int_{-\infty}^{\infty} f(y)e^{-y^2}e^{2xy}dy=0$$

for all $x\in\mathbb{R}$. Prove that $f(x)=0$.

PROBLEM 6

Let $(X,\mathfrak{M},\mu)$ be a sigma-finite measure space, let $f:X\to\mathbb{R}_+$ be a measurable function, and let $\phi:\mathbb{R}_+\to\mathbb{R}_+$ be an increasing function. Here $\mathbb{R}_+=\{a\in\mathbb{R}: a\geq 0\}$, and the measure on $\mathbb{R}_+$ is the Lebesgue measure. Let

$$F(t)=\mu(\{x: f(x)>t\}).$$

Prove that

$$\int_X \phi(f(x))d\mu=\int_0^{\infty} F\big(\phi^{-1}(\tau)\big)d\tau.$$

Hint. The Fubini–Tonelli Theorem may be of help.