ANALYSIS QUALIFYING EXAM
AUGUST, 2008

PROBLEM 1
Find all pairs of positive numbers $\alpha$ and $\beta$ for which the function $x^\alpha \cos(x^{-\beta})$ is of bounded variation on the interval $[0, 1]$.

PROBLEM 2
Find the limit
$$\lim_{n \to \infty} n \int_0^\infty \frac{\arctan x}{1 + n^2 x^2} \, dx$$
Justify all steps.

PROBLEM 3
Let $F : [0, 1]^2 \to [0, 1]$ be a mapping given by the formula $F(x, y) = xy$. Let $\mathcal{B}[0, 1]$ be the Borel sigma algebra on $[0, 1]$, and let $\mathcal{M}_F = \{F^{-1}(B) : B \in \mathcal{B}[0, 1]\}$. Show that $\mathcal{M}_F$ is a sigma algebra of subsets of $[0, 1]^2$. Describe functions $f(x, y)$ that are $\mathcal{M}_F$-measurable.

PROBLEM 4
Let $f(x) \in L^1([0, 1])$. Suppose that
$$\int_a^b f(x) \, dx = (b - a) \int_a^1 f(x) \, dx$$
for all rational numbers $a$ and $b$. What can be said about the function $f(x)$?

PROBLEM 5
Let $f(x) \in L^2((-\infty, \infty))$. Suppose that
$$\int_{-\infty}^\infty f(y) e^{-y^2} e^{2xy} \, dy = 0$$
for all $x \in \mathbb{R}$. Prove that $f(x) = 0$.

PROBLEM 6
Let $(X, \mathcal{X}, \mu)$ be a sigma-finite measure space, let $f : X \to \mathbb{R}_+$ be a measurable function, and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function. Here $\mathbb{R}_+ = \{a \in \mathbb{R} : a \geq 0\}$, and the measure on $\mathbb{R}_+$ is the Lebesgue measure. Let $F(t) = \mu(\{x : f(x) > t\})$.

Prove that
$$\int_X \phi(f(x)) \, d\mu = \int_0^\infty F(\phi^{-1}(r)) \, dr.$$

Hint. The Fubini–Tonelli Theorem may be of help.