Analysis qualifying exam - August 2012

Please show all your work. GOOD LUCK!

(1) $x_0 \in \mathbb{R}$ is a zero of a continuous function $f$ if $f(x_0) = 0$. $x_0$ is an isolated zero of $f$ if there exists an open set containing $x_0$, which contains no other zeros of $f$.

(a) Show that there exists a continuous function $f : (0, 1) \to \mathbb{R}$ with infinitely many isolated zeros.

(b) Show that if $f : [0, 1] \to \mathbb{R}$ is continuous and all of its zeros are isolated, then it only has finitely many zeros.

(2) Let $L^p(\mathbb{R}, dx)$ be the usual $L^p$ space on the real line with respect to Lebesgue measure. Let $p, q > 1$ and suppose $f \in L^p(\mathbb{R}, dx)$ and $g \in L^q(\mathbb{R}, dx)$. Show that if $1 > 1/p + 1/q > 1/2$, then
\[
\frac{f(x)g(x)}{1 + \sqrt{|x|}} \in L^1(\mathbb{R}, dx)
\]

(3) $E_n$ is a sequence of measurable sets in a finite measure space $(X, \mathcal{M}, \mu)$. We define the sets
\[
\limsup E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k, \quad \liminf E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k,
\]
Show that the sequence of characteristic functions $\chi_{E_n}$ converges in $L^2(\mu)$ if and only if $\mu(\limsup E_n \setminus \liminf E_n) = 0$.

(4) Compute, with justification, the value of the following limit.
\[
\lim_{n \to \infty} n \int_0^\infty \sin(x) e^{-nx^2} \, dx.
\]
(Hint: It might be useful to make a change variables to $y = n^\beta x$ for an appropriate exponent $\beta$.)

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(5) Let \((X, \mathcal{M}, \mu)\) be a finite measure space. Let \(g(t)\) be a real-valued function on \(\mathbb{R}\) which is continuously differentiable, has \(g'(t) \leq 0\), and \(\lim_{t \to \infty} g(t) = 0\). Let \(f\) be measurable on \(X\) and define \(F(t) = \mu(\{x : f(x) \leq t\})\). Show that if
\[
\int_{-\infty}^{\infty} |g'(t)| F(t) \, dt < \infty
\]
then \(g(f(x))\) is integrable with respect to \(\mu\) and
\[
-\int_{-\infty}^{\infty} g'(t) F(t) \, dt = \int_X g(f(x)) \, d\mu
\]
Hint: express \(F(t)\) as an integral over \(X\).

(6) \(\mathcal{H}\) is a separable Hilbert space and \(\{\phi_k\}_{k \in \mathbb{N}}\) is an orthonormal basis for \(\mathcal{H}\). A sequence \(x_n\) in \(\mathcal{H}\) is unbounded if the sequence \(\|x_n\|\) is unbounded in \(\mathbb{R}\).

(a) Give an example of an unbounded sequence \(x_n \in \mathcal{H}\) such that \(\langle x_n, \phi_k \rangle \to 0\) for all \(k \in \mathbb{N}\).

(b) Show that there is a dense set \(E \subseteq \mathcal{H}\) such that \(\langle x_n, z \rangle \to 0\) for all \(z \in E\), where \(x_n\) is the sequence from part (a).

(c) If \(x_n\) is an unbounded sequence in \(\mathcal{H}\), show that there is a \(y \in \mathcal{H}\) such that the sequence \(\langle x_n, y \rangle\) is unbounded.