Analysis qualifying exam - August 2012

Please show all your work. GOOD LUCK!

(1) $x_0 \in \mathbb{R}$ is a zero of a continuous function f if $f(x_0) = 0$. x_0 is an *isolated* zero of f if there exists an open set containing x_0, which contains no other zeros of f.

(a) Show that there exists a continuous function $f : (0,1) \to \mathbb{R}$ with infinitely many isolated zeros.

(b) Show that if $f : [0,1] \to \mathbb{R}$ is continuous and all of its zeros are isolated, then it only has finitely many zeros.

(2) Let $L^p(\mathbb{R}, dx)$ be the usual L^p space on the real line with respect to Lebesgue measure. Let $p, q > 1$ and suppose $f \in L^p(\mathbb{R}, dx)$ and $g \in L^q(\mathbb{R}, dx)$. Show that if $1 > 1/p + 1/q > 1/2$, then

$$\frac{f(x)g(x)}{1 + \sqrt{|x|}} \in L^1(\mathbb{R}, dx)$$

(3) E_n is a sequence of measurable sets in a finite measure space $(X, \mathcal{M}, \mu)$. We define the sets

$$\limsup E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k, \qquad \liminf E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k,$$

Show that the sequence of characteristic functions χ_{E_n} converges in $L^2(\mu)$ if and only if $\mu(\limsup E_n \setminus \liminf E_n) = 0$.

(4) Compute, with justification, the value of the following limit.

$$\lim_{n\to\infty} n \int_0^{\infty} \sin(x)\, e^{-nx^2}\, dx.$$

(Hint: It might be useful to make a change variables to $y = n^{\beta} x$ for an appropriate exponent β.)

(5) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $g(t)$ be a real-valued function on $\mathbb{R}$ which is continuously differentiable, has $g'(t) \leq 0$, and $\lim_{t\to\infty} g(t) = 0$. Let f be measurable on X and define $F(t) = \mu(\{x : f(x) \leq t\})$. Show that if

$$\int_{-\infty}^{\infty} |g'(t)|\, F(t)\, dt < \infty$$

then $g(f(x))$ is integrable with respect to μ and

$$-\int_{-\infty}^{\infty} g'(t)\, F(t)\, dt = \int_X g(f(x))\, d\mu$$

Hint: express $F(t)$ as an integral over X.

(6) $\mathcal{H}$ is a separable Hilbert space and $\{\phi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$. A sequence x_n in $\mathcal{H}$ is unbounded if the sequence $\|x_n\|$ is unbounded in $\mathbb{R}$.

(a) Give an example of an unbounded sequence $x_n \in \mathcal{H}$ such that $\langle x_n, \phi_k\rangle \to 0$ for all $k \in \mathbb{N}$.

(b) Show that there is a dense set $E \subseteq \mathcal{H}$ such that $\langle x_n, z\rangle \to 0$ for all $z \in E$, where x_n is the sequence from part (a).

(c) If x_n is an unbounded sequence in $\mathcal{H}$, show that there is a $y \in \mathcal{H}$ such that the sequence $\langle x_n, y\rangle$ is unbounded.