ANALYSIS QUALIFYING EXAM

FALL 2017

Please show all of your work. GOOD LUCK!

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. For any $n \geq 1$, take $f_n : \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) = \int_0^1 f\left(x + \frac{y}{n}\right) dy$$

a) Find $\lim_{n\to\infty} f_n(x)$ and show that f_n converges uniformly to this function on every compact interval of $\mathbb{R}$.

b) Show, by counterexample, that this convergence need not be uniform on the entire real line.

(2) a) Let $A \subset [0, 1]$ be a set of Lebesgue measure 0. Show that $B := \{x^2 \, : \, x \in A\}$ also has Lebesgue measure 0.

b) Let $g : [0, 1] \to \mathbb{R}$ be an absolutely continuous, monotone function, and let $A \subset [0, 1]$ be a set of Lebesgue measure 0. Show that $g(A)$ also has Lebesgue measure 0.

(3) Let p and q be conjugate exponents with $1 < p, q < \infty$. For any $f \in L^p([0, 1])$ set

$$g(x) = \int_0^x f(t)\, dt\,.$$

Show that $g \in L^q([0, 1])$ and prove that

$$\|g\|_q \leq 2^{-1/q}\|f\|_p\,.$$

(4) Let μ be a finite Borel measure on $\mathbb{R}$. For any $f \in L^1(\mu)$, compute, with justification,

$$\lim_{n\to\infty} \int_0^1 |f(x)|^{1/n} d\mu(x).$$

(5) Let $I = [-\pi, \pi]$. For each $n \geq 1$, let $f_n : I \to \mathbb{R}$ be given by

$$f_n(x) = c_n \sin(nx) \quad \text{with} \quad c_n \in \mathbb{C}\,.$$

a) Find necessary and sufficient conditions on the coefficients c_n for the sequence f_n to converge to zero in $L^2(I)$.
b) Find necessary and sufficient conditions on the coefficients c_n for the sequence f_n to weakly converge to zero in $L^2(I)$.
Recall: f_n converges weakly to f in a Hilbert space $\mathcal{H}$ if $\langle g, f_n \rangle \to \langle g, f \rangle$ for all $g \in \mathcal{H}$.

(6) Consider the function

$$I(f) \;=\; \int_0^1 f(x) \ln f(x) dx$$

defined on the set of non-negative $L^1([0,1])$ functions; here the integral is with respect to Lebesgue measure on the unit interval $[0,1]$.
a) Show that I is convex. **Hint:** It may be useful to consider the function $\phi : [0,1] \to \mathbb{R}$ given by $\phi(t) = t\ln(t)$ with $\phi(0) = 0$.
b) Show that I is lower semi-continuous, i.e.

$$I(f) \leq \liminf_{n\to\infty} I(f_n)$$

whenever f_n is a sequence of non-negative functions converging in $L^1([0,1])$ to f. **Hint:** $\phi(t)$ is bounded below.