ANALYSIS QUALIFYING EXAM

Fall 2017

Please show all of your work. GOOD LUCK!

1) Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. For any \( n \geq 1 \), take \( f_n : \mathbb{R} \to \mathbb{R} \) as

\[
f_n(x) = \int_0^1 f \left( x + \frac{y}{n} \right) dy
\]

a) Find \( \lim_{n \to \infty} f_n(x) \) and show that \( f_n \) converges uniformly to this function on every compact interval of \( \mathbb{R} \).

b) Show, by counterexample, that this convergence need not be uniform on the entire real line.

2) a) Let \( A \subset [0, 1] \) be a set of Lebesgue measure 0. Show that \( B := \{ x^2 : x \in A \} \) also has Lebesgue measure 0.

b) Let \( g : [0, 1] \to \mathbb{R} \) be an absolutely continuous, monotone function, and let \( A \subset [0, 1] \) be a set of Lebesgue measure 0. Show that \( g(A) \) also has Lebesgue measure 0.

3) Let \( p \) and \( q \) be conjugate exponents with \( 1 < p, q < \infty \). For any \( f \in L^p([0, 1]) \) set

\[
g(x) = \int_0^x f(t) \, dt.
\]

Show that \( g \in L^q([0, 1]) \) and prove that

\[
\|g\|_q \leq 2^{-1/q} \|f\|_p.
\]

4) Let \( \mu \) be a finite Borel measure on \( \mathbb{R} \). For any \( f \in L^1(\mu) \), compute, with justification,

\[
\lim_{n \to \infty} \int_0^1 |f(x)|^{1/n} d\mu(x).
\]
(5) Let \( I = [-\pi, \pi] \). For each \( n \geq 1 \), let \( f_n : I \to \mathbb{R} \) be given by
\[
f_n(x) = c_n \sin(nx) \quad \text{with} \quad c_n \in \mathbb{C}.
\]
a) Find necessary and sufficient conditions on the coefficients \( c_n \) for the sequence \( f_n \) to converge to zero in \( L^2(I) \).
b) Find necessary and sufficient conditions on the coefficients \( c_n \) for the sequence \( f_n \) to weakly converge to zero in \( L^2(I) \).
\textbf{Recall:} \( f_n \) converges weakly to \( f \) in a Hilbert space \( \mathcal{H} \) if \( \langle g, f_n \rangle \to \langle g, f \rangle \) for all \( g \in \mathcal{H} \).

(6) Consider the function
\[
I(f) = \int_0^1 f(x) \ln f(x) dx
\]
defined on the set of non-negative \( L^1([0, 1]) \) functions; here the integral is with respect to Lebesgue measure on the unit interval \([0, 1] \).
a) Show that \( I \) is convex. \textbf{Hint:} It may be useful to consider the function \( \phi : [0, 1] \to \mathbb{R} \) given by \( \phi(t) = t \ln(t) \) with \( \phi(0) = 0 \).
b) Show that \( I \) is lower semi-continuous, i.e.
\[
I(f) \leq \liminf_{n \to \infty} I(f_n)
\]
whenever \( f_n \) is a sequence of non-negative functions converging in \( L^1([0, 1]) \) to \( f \). \textbf{Hint:} \( \phi(t) \) is bounded below.