Real Analysis Qualifying Exam, January 2008

1. Let $f \geq 0$ be a real measurable function on $[0,1]$. Show that

$$\left(\int_0^1 \sin(x) f(x)\, dx\right)^4 \leq \int_0^1 f(x)^4\, dx.$$

2. Let $f(x)$ be a continuously differentiable function on $[0,1]$. For $\alpha > 0$, we define the set

$$S_\alpha = \{x : |f'(x)| < \alpha\}.$$

Prove that the Lebesgue measure of the image satisfies

$$\text{meas} f(S_\alpha) \leq \alpha.$$

3. Let $f : [0,1] \to \mathbf{R}$ be a measurable function. Let $A = \{x \in [0,1] \mid f(x) \in \mathbf{Z}\}$, where $\mathbf{Z}$ is the set of integers. Prove that

$$\int_0^1 \cos(\pi(f(x)))^{2n}\, dx \to \lambda(A)$$

as $n \to \infty$, where $\lambda(A)$ is the Lebesgue measure of A.

4. Let $f(x)$ be an L^2 function function on $(-\infty, \infty)$. For $a > 0$, we define the function

$$f_a(x) = \frac{1}{2a} \int_{(x-a,x+a)} f(y) dy.$$

a) Prove that for each $a > 0$ the function f_a is continuous.
b) Prove that f_a converges to f in the L^2 sense as $a \to 0$.

5. Let $\alpha > 0$. For what real values of λ does the function

$$f(x,y) = \big||x|^\alpha - |y|^\alpha\big|^\lambda$$

on the disk $x^2 + y^2 \leq 1$ belong to L^1 with respect to two-dimensional Lebesgue measure?

6. Let X be the metric space consisting of all nonempty closed subsets of $[0,1]$ with the metric

$$\delta(F,G) = \max\{\sup_{x \in F} d(x,G), \sup_{y \in G} d(y,F)\}.$$

a) Prove that X is totally bounded.

b) Prove that X is complete. Hint: Let F_n be a Cauchy sequence. Define F by saying that z is in F if for every $\epsilon > 0$, there are infinitely many m such that F_m has non-empty intersection with the ϵ ball about z. Prove that $\delta(F_n, F) \to 0$ as $n \to \infty$.