ANALYSIS QUALIFYING EXAM

JANUARY 2010

Please show all of your work. GOOD LUCK!

(1) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function of compact support. Calculate

$$\lim_{\epsilon \to 0^+} \operatorname{Im} \left[\int_{\mathbb{R}} \frac{1}{x - i\epsilon} \phi(x)\, dx \right].$$

(2) Calculate

$$\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{-x^2} \cos(nx)\, dx\,.$$

(3) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $0 < p < \infty$ and consider $f \in L^p(X, d\mu)$. Prove that

$$\mu\left(\{x \in X : |f(x)| > t\}\right) \leq \frac{1}{t^p} \int_X |f(x)|^p d\mu(x)$$

for all $t > 0$.

(4) Let $f, g \geq 0$ be bounded, measurable functions on $[0, 1]$. Take real $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that the function

$$t \mapsto \ln \left[\int_0^1 f^{pt}(x) g^{q(1-t)}(x) dx \right]$$

is a convex function of t.

(5) Let $f : [a, b] \to \mathbb{R}$ be differentiable at each $x \in [a, b]$. Let λ be a real number with $f'(a) < \lambda < f'(b)$. Prove that there exists $y \in (a, b)$ with $f'(y) = \lambda$. **Hint:** It may be helpful to consider the function $g : [a, b] \to \mathbb{R}$ defined by setting $g(t) = f(t) - \lambda t$.

(6) The convolution of two functions f and g is defined to be

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y)dy$$

when the above integral exists.
a) Prove that if f and g are in $L^1(\mathbb{R}, dx)$ then $f * g$ exists for almost every $x \in \mathbb{R}$ and

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1 .$$

b) Let $1 \le p < \infty$, prove that if $f \in L^1(\mathbb{R}, dx)$ and $g \in L^p(\mathbb{R}, dx)$, then $f * g$ exists for almost every $x \in \mathbb{R}$ and

$$\|f * g\|_p \le \|f\|_1 \|g\|_p .$$

Hint: For part b) use the argument in a) and Jensen's inequality.