ANALYSIS QUALIFYING EXAM

PLEASE SHOW ALL YOUR WORK

GOOD LUCK!

PROBLEM 1

Let \( p \) and \( q \) be positive numbers. Prove that the integral

\[
\int_0^1 \int_0^1 \frac{dxdy}{x^p + y^q}
\]

converges if and only if \( p^{-1} + q^{-1} > 1 \).

PROBLEM 2

Find the following limit:

\[
\lim_{n \to \infty} \int_0^\infty \frac{dx}{(1 + \frac{x}{n})^{x^{1/n}}}
\]

Justify all steps.

PROBLEM 3

Let \( f : [a, b] \to \mathbb{R} \) be an integrable function. Prove that

\[
\left( \int_a^b f(x) \sqrt{|\cos x|} dx \right)^4 + \left( \int_a^b f(x) \sqrt{|\sin x|} dx \right)^4 \leq \left( \int_a^b |f(x)| dx \right)^4.
\]

PROBLEM 4

Let \( f_n(x) \) be a sequence of differentiable functions on the interval \([0, 1]\). Suppose that \( f_n(x) \to 0 \) pointwise and that \( |f_n'(x)| \leq M \) for some constant \( M \) that is independent of \( n \) and \( x \). Prove that \( f_n(x) \to 0 \) uniformly.

PROBLEM 5

Let \( S^1 \) be a circle of length \( 2\pi \). We denote a coordinate on \( S^1 \) by \( \theta \). As usual, \( \theta \) and \( \theta + 2\pi \) correspond to the same point on \( S^1 \). Let \( m = d\theta \) be the Lebesgue measure on \( S^1 \). Let \( M \subset S^1 \) be a measurable set such that \( m(M) \geq \frac{3\pi}{2} \). Let

\[
X = \{ \theta \in S^1 : m(M \cap (\theta - 0.1, \theta + 0.1)) \leq 0.1 \}.
\]

Prove that \( m(X) \leq \pi \).

Hint. You may consider the intersection of the sets \( \{ (\omega, \theta) \in S^1 \times S^1 : |\omega - \theta| < 0.1 \} \) and \( M \times S^1 \).

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Problem 6

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f \in L^2(X, \mathcal{M}, \mu)$. Prove that there exist functions $g \in L^1(X, \mathcal{M}, \mu)$ and $h \in L^\infty(X, \mathcal{M}, \mu)$ such that $f = g + h$ and $\|g\|_1 + \|h\|_{\infty} \leq 2\|f\|_2$. 