Analysis qualifying exam - January 2013

(1) Let (X,μ) be a measure space. Fix a function $g \in L^3(X,\mu)$.

(a) Prove that if $f \in L^6(X,\mu)$, then fg is in $L^2(X,\mu)$.

(b) Define $T : L^6(X,\mu) \to L^2(X,\mu)$ by $T(f) = fg$. Prove that T is continuous.

(2) Consider $C([0,1])$, the set of real-valued continuous functions on $[0,1]$, BUT give it the topology that comes from the L^1 norm:

$$||f||_1 = \int_0^1 |f(x)|\,dx$$

Fix a nonnegative continuous function $g(x)$. Let

$$E = \{f \in C([0,1]) : 0 \le f(x) \le g(x), \forall x \in [0,1]\}$$

Prove that E is closed.

(3) Let (X,μ) be a measure space. Let f be an integrable function with $f > 0$ a.e. For $t \ge 0$ define

$$F(t) = \int_{E_t} f(x) d\mu(x)$$

where $E_t = \{x : f(x) \ge t\}$. Define $B_t = \{x : f(x) = t\}$. Prove that $F(t)$ is continuous at t_0 if and only if $\mu(B_{t_0}) = 0$.

(4) H is an infinite dimensional Hilbert space over $\mathbb{C}$.

(a) $\{b_n\}$ is a complex sequence. $\sum_n \bar{a}_n b_n$ converges for all ℓ^2 sequences a_n. Prove that $\sum_n |b_n|^2 < \infty$.

(b) $\{\mathbf{x}_n\}$ is a sequence of elements in H such that $\sum_n \|\mathbf{x}_n\|_H^2 < \infty$. Prove that $\sum_n a_n \mathbf{x}_n$ converges for all ℓ^2 sequences a_n.

(c) Show that there exists a sequence $\{\mathbf{x}_n\}$ of elements in H such that $\sum_n \|\mathbf{x}_n\|_H^2$ diverges, but $\sum_n a_n \mathbf{x}_n$ converges for all ℓ^2 sequences a_n.

(5) Let $X = L^1(\mathbb{R})$ where $\mathbb{R}$ is equipped with Lebesgue measure. Let $g \in X$ be a given function.

(a) Show that the mapping L_g defined by $L_g(f) = g * f$ where

$$g * f(x) = \int g(y) f(x-y) dy$$

defines a bounded linear map from X into itself.

(b) Consider the sequence of L^1 functions

$$e_n(x) = \begin{cases} n & |x| \leq \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

Show that $\liminf_{n\to\infty} \|L_g(e_n)\|_X \geq \|g\|_X$.

(c) Find the induced norm of the mapping $L_g : X \to X$.

(6) Let (X, μ) be a finite measure space. Let $f \geq 0$ be a measurable function on X. Define

$$c_n = \mu(\{x : 2^n \leq f(x) < 2^{n+1}\})$$

Let R be the radius of convergence of the series $\sum_{n=0}^{\infty} c_n z^n$. Also, for $p > 0$, we say $f \in L^p(\mu)$ if $\int_X f^p \, d\mu < \infty$.

(a) Show that $R \geq 1$, and $R = 1$ if and only if $f \notin L^p(\mu)$ for any $p > 0$.

(b) Let

$$\rho = \sup\{p > 0 : f \in L^p(\mu)\}$$

with the convention that the supremum of the empty set is $-\infty$. Express ρ in terms of R and prove your answer.