Geometry-Topology Qualifying Exam. August 2006.

Solve only one of the following two problems.

1A. Compute the following integral:

$$\int_0^\infty \frac{\cos(x)}{1+x^4}\, dx.$$

1B. Find a conformal mapping of the vertical semi-infinite strip $\{0 < \mathrm{Re}(z) < 1,\ \mathrm{Im}(z) > 0\}$ onto the unit disc $|w| < 1$.

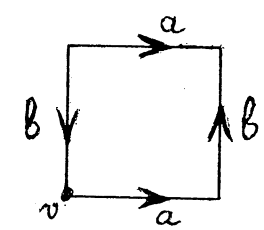
2. Compute the singular homology groups $H_*(X, \mathbb{Z})$ of the space $X = \mathbb{R}^3 \setminus A$, where A is a subset of $\mathbb{R}^3$ homeomorphic to the disjoint union of two unlinked circles.

3. Consider the following map $f : \mathbb{R}^3 \to \mathbb{R}^2$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} xz - y^2 \\ yz - x^2 \end{pmatrix} .$$

For which values $(a, b) \in \mathbb{R}^2$ of f is the level set $f^{-1}(a, b)$ a smooth submanifold of $\mathbb{R}^3$?

4. Consider the surface Σ obtained by identifying the edges of a square in the following way:



(a) Construct a model of the universal covering space of this surface, indicating especially how $\pi_1(\Sigma, v)$ acts.
(b) Identify the covering space X of Σ, which corresponds to the subgroup of $\pi_1(\Sigma, v)$ generated by a and describe the group of covering automorphisms of X.

5. Consider the submanifold $\iota : M \hookrightarrow \mathbb{R}^3$ given by $x^2 + y^2 - z^2 = 1$.
(a) Show that the vector field $X = \dfrac{xz}{1+z^2}\dfrac{\partial}{\partial x} + \dfrac{yz}{1+z^2}\dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}$ is tangent to M, i.e. that there exists a vector field Y on M such that for any $m \in M$ we have $\iota_*(Y(m)) = X(m)$.
(b) Show that the two-form $\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$ restricts to an area form on M, i.e. a two-form which never vanishes. (*Hint*: use cylindrical coordinates.)
(c) Does the flow of Y on M preserve $\iota^*(\omega)$?

6. Prove the Poincaré lemma in the plane: a closed 1-form or 2-form on $\mathbb{R}^2$ is exact.

7. Let T^2 be the two-dimensional torus and let $\phi :\ S^2 \to T^2$ be a smooth map. Show that for any top de Rham cohomology class $[\nu] \in H^2(T^2)$, we have $\phi^*[\nu] = 0$.