GEOMETRY-TOPOLOGY EXAM. AUGUST 2007.

1. Use contour integration to evaluate

$$\int_0^{2\pi} (\cos\theta)^n \, d\theta, \quad \text{for all } \ n \geq 0.$$

2. Let $\mathrm{x} = [x_0, x_1, \ldots, x_m]$ be homogeneous coordinates on $\mathbb{RP}^m$, which determine the line $(tx_0, tx_1, \ldots, tx_m)$, $t \in \mathbb{R}$ that represents a point of $\mathbb{RP}^m$. Let $\mathrm{y} = [y_0, y_1, \ldots, y_n]$ be homogeneous coordinates on $\mathbb{RP}^n$. Suppose that $m \leq n$. Show that the set

$$\{(\mathrm{x}, \mathrm{y}) \mid \sum_{j=0}^{m} x_j y_j = 0\} \subset \mathbb{RP}^m \times \mathbb{RP}^n$$

is an embedded submanifold of $\mathbb{RP}^m \times \mathbb{RP}^n$, and determine its dimension.

3. Let $\mathbb{T}$ be the solid torus in $\mathbb{R}^3$, which is obtained by revolving the disc $(x-2)^2 + z^2 \leq 1$ in the xz-plane around the z-axis. Compute the homology groups of the space $X = \mathbb{T}/\sim$ obtained by identifying the pairs of points on the boundary of $\mathbb{T}$, which are symmetric about the origin, i.e. $(x, y, z) \sim (-x, -y, -z)$ for $(x, y, z) \in \partial(\mathbb{T})$.

4. Let $\iota : \mathbb{S}^3 \hookrightarrow \mathbb{R}^4$ be the inclusion map of the unit sphere and consider the following 3-form on $\mathbb{R}^4$:

$$\alpha = x_1 \, dx_2 \wedge dx_3 \wedge dx_4 - x_2 \, dx_1 \wedge dx_3 \wedge dx_4 + x_3 \, dx_1 \wedge dx_2 \wedge dx_4 - x_4 \, dx_1 \wedge dx_2 \wedge dx_3 \ .$$

Also let $\beta = \iota^*(\alpha)$.

(1) Are either α or β exact and/or closed ?

(2) Evaluate $\displaystyle\int_{\mathbb{S}^3} \beta$.

(3) Let γ be the following 3-form on $\mathbb{R}^4 \setminus \{0\}$:

$$\gamma = \frac{\alpha}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^k} \ , \quad k \in \mathbb{R} \ .$$

Determine the values of k for which γ is closed and those for which it is exact.

5. Prove that there is no such continuous map $f : \mathbb{S}^2 \to \mathbb{S}^1$, that satisfies $f \circ \alpha = \beta \circ f$, where α and β are the antipodal maps on $\mathbb{S}^2$ and $\mathbb{S}^1$ respectively.

6. Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$, given by $x^2+y^2+z^2=1$. Let U be the coordinate chart $U := \{(x,y,z) \mid 0 \le x^2+z^2 < 1,\ y > 0\}$ with local coordinates (x,z). A certain vector field X on $\mathbb{S}^2$ has the form

$$\mathrm{X}\big|_U = \sqrt{1-x^2-z^2}\,\frac{\partial}{\partial x}$$

in the coordinate chart U.

a) Sketch the vector field $\mathrm{X}\big|_U$ in the local coordinates, i.e. in the domain $0 \le x^2+z^2 < 1$, find the integral curves $(x(t), z(t))$ explicitly, and sketch them.

b) Define a coordinate chart V that contains the point $(x,y,z) = (0,0,1)$, and define suitable local coordinates (ξ, η). Express X in $U \cap V$ in terms of the your local coordinates (ξ, η).

c) Sketch the vector field $\mathrm{X}\big|_{U\cap V}$ in your local coordinates (ξ, η), find the integral curves $(\xi(t), \eta(t))$ explicitly, and sketch them.

d) Give a geometric interpretation of X on $\mathbb{S}^2 \subset \mathbb{R}^3$.

7. Let a and b be the generators of $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ corresponding to the two $\mathbb{S}^1$ summands. Describe the covering space of $\mathbb{S}^1 \vee \mathbb{S}^1$ corresponding to the subgroup generated by a^2, b, and aba^{-1}, and determine the group of its deck transformations.