1. Use contour integration to evaluate
\[ \int_0^{2\pi} (\cos \theta)^n \, d\theta, \quad \text{for all} \quad n \geq 0. \]

2. Let \( x = [x_0, x_1, \ldots, x_m] \) be homogeneous coordinates on \( \mathbb{RP}^m \), which determine the line \( (tx_0, tx_1, \ldots, tx_m), \ t \in \mathbb{R} \) that represents a point of \( \mathbb{RP}^m \). Let \( y = [y_0, y_1, \ldots, y_n] \) be homogeneous coordinates on \( \mathbb{RP}^n \). Suppose that \( m \leq n \). Show that the set
\[ \{(x, y) \mid \sum_{j=0}^m x_j y_j = 0\} \subseteq \mathbb{RP}^m \times \mathbb{RP}^n \]
is an embedded submanifold of \( \mathbb{RP}^m \times \mathbb{RP}^n \), and determine its dimension.

3. Let \( T \) be the solid torus in \( \mathbb{R}^3 \), which is obtained by revolving the disc \((x - 2)^2 + z^2 \leq 1\) in the \(xz\)-plane around the \(z\)-axis. Compute the homology groups of the space \( X = T/\sim \) obtained by identifying the pairs of points on the boundary of \( T \), which are symmetric about the origin, i.e. \((x, y, z) \sim (-x, -y, -z)\) for \((x, y, z) \in \partial(T)\).

4. Let \( i : S^3 \hookrightarrow \mathbb{R}^4 \) be the inclusion map of the unit sphere and consider the following 3-form on \( \mathbb{R}^4 \):
\[ \alpha = x_1 \, dx_2 \wedge dx_3 \wedge dx_4 - x_2 \, dx_1 \wedge dx_3 \wedge dx_4 + x_3 \, dx_1 \wedge dx_2 \wedge dx_4 - x_4 \, dx_1 \wedge dx_2 \wedge dx_3. \]
Also let \( \beta = i^*(\alpha) \).

(1) Are either \( \alpha \) or \( \beta \) exact and/or closed?
(2) Evaluate \( \int_{S^3} \beta \).
(3) Let \( \gamma \) be the following 3-form on \( \mathbb{R}^4 \setminus \{0\} \):
\[ \gamma = \frac{\alpha}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^k}, \quad k \in \mathbb{R}. \]
Determine the values of \( k \) for which \( \gamma \) is closed and those for which it is exact.
5. Prove that there is no such continuous map \( f : \mathbb{S}^2 \to \mathbb{S}^1 \), that satisfies \( f \circ \alpha = \beta \circ f \), where \( \alpha \) and \( \beta \) are the antipodal maps on \( \mathbb{S}^2 \) and \( \mathbb{S}^1 \) respectively.

6. Let \( \mathbb{S}^2 \) be the unit sphere in \( \mathbb{R}^3 \), given by \( x^2 + y^2 + z^2 = 1 \). Let \( U \) be the coordinate chart \( U := \{ (x, y, z) \mid 0 \leq x^2 + y^2 < 1, \ y > 0 \} \) with local coordinates \( (x, z) \). A certain vector field \( \mathbf{X} \) on \( \mathbb{S}^2 \) has the form

\[
\mathbf{X}|_U = \sqrt{1 - x^2 - z^2} \frac{\partial}{\partial x}
\]

in the coordinate chart \( U \).

a) Sketch the vector field \( \mathbf{X}|_U \) in the local coordinates, i.e. in the domain \( 0 \leq x^2 + z^2 < 1 \), find the integral curves \( (x(t), z(t)) \) explicitly, and sketch them.

b) Define a coordinate chart \( V \) that contains the point \( (x, y, z) = (0, 0, 1) \), and define suitable local coordinates \( (\xi, \eta) \). Express \( \mathbf{X} \) in \( U \cap V \) in terms of the your local coordinates \( (\xi, \eta) \).

c) Sketch the vector field \( \mathbf{X}|_{U \cap V} \) in your local coordinates \( (\xi, \eta) \), find the integral curves \( (\xi(t), \eta(t)) \) explicitly, and sketch them.

d) Give a geometric interpretation of \( \mathbf{X} \) on \( \mathbb{S}^2 \subset \mathbb{R}^3 \).

7. Let \( a \) and \( b \) be the generators of \( \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) \) corresponding to the two \( \mathbb{S}^1 \) summands. Describe the covering space of \( \mathbb{S}^1 \vee \mathbb{S}^1 \) corresponding to the subgroup generated by \( a^2, b, \) and \( aba^{-1} \), and determine the group of its deck transformations.