GEOMETRY-TOPOLOGY PH.D. QUALIFYING EXAM, AUGUST 2008

Problem 4 counts 20 points, and the others count 10 points each. The perfect score is 70.

1. Suppose that $0 \leq b < 1$ and $p > 0$. Calculate the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+2bx+x^2} e^{ixp}\, dx.$$

[The emphasis in this problem is on the calculation; you do not need to justify the steps in your calculation]

2. (a) Determine the values of a for which

$$X_a = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = a\}$$

is an embedded submanifold of $\mathbb{R}^3$.

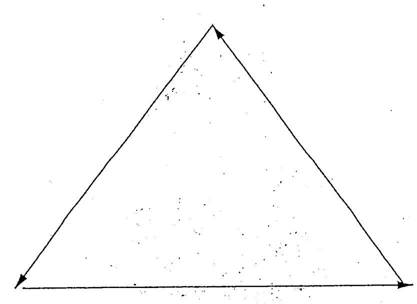
(b) Suppose that $a > 0$, and let $X_a^+ = \{(x,y,z) \in X_a : z > 0\}$ with orientation induced by the upward pointing normal vector. Note that there is a global coordinate chart $\phi : X_a^+ \to \mathbb{R}^2$, $\phi : (x,y,z) \mapsto (x,y)$. Calculate the area form dA in this coordinate.

(c) Determine the values of p for which the integral

$$\int_{X_a^+} z^{-p} dA$$

is finite.

3. Compute the fundamental group and the homology groups of the triangle with three sides identified according to the arrows shown in the figure. Show the details.



4. In each part, determine whether there exists an example with the given property. If yes, give an example and explain why the desired conditions are satisfied. If no, explain why there cannot be an example.

a) a compact two-manifold X for which $H_2(X, \mathbb{Z}) = 0$ (the boundary of X is empty);
b) a one-form η on $\mathbb{R}^2 \setminus \{0\}$ such that

$$\int_{S^1} \eta \neq 0;$$

c) an integer n for which the product $\mathbb{RP}^3 \times \mathbb{RP}^n$ is orientable;
d) a nonvanishing exact k-form on a compact orientable k-dimensional manifold;
e) a nonnormal covering space of the figure eight; .
f) a space which has $F_2 \times (\mathbb{Z}/2\mathbb{Z})$ (the direct product) as fundamental group, where F_2 is the free group on two generators;
g) a smooth mapping $f : M \to N$ of smooth manifolds that is one-to-one and onto, but is not a diffeomorphism.

5. Set

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, y > 0\}.$$

Define a one-form ω on M by $\omega = dz + x\,dy - y\,dx$. Define the *kernel* of ω to be the linear space $\mathcal{D}$ of vector fields Z on M for which $\omega(Z) = 0$.

a) Find vector fields X, Y in $\mathcal{D}$ such that every vector field $Z \in \mathcal{D}$ can be written in the form $aX + bY$ with appropriate smooth functions a and b. (One says that X and Y are a basis, over $C^\infty(M)$, of $\mathcal{D} = \ker \omega$). There is not a unique choice of X, Y.
b) For $p \in M$, let X_p, Y_p denote the values of your vector fields X, Y at p, and let $\mathcal{D}_p$ be their span. Show that there is no smooth function $f : M \to \mathbb{R}$ for which these spaces $\mathcal{D}_p$ coincide with the tangent spaces to the level sets $f^{-1}(c)$.

   [More precisely. Suppose that all the sets $N_c := f^{-1}(c)$ are smooth manifolds. Look at the tangent bundle TN_c of such a level set. It is not possible that $T_pN_c = \mathcal{D}_p$ for all $p \in N_c$.]

6. Let M be a smooth manifold, and let $\mathcal{A}^p(M)$ be the space of smooth p-forms. Define a map F by

$$F : \mathcal{A}^p(M) \times \mathcal{A}^q(M) \to \mathcal{A}^{p+q}(M), \quad (\eta, \omega) \mapsto \eta \wedge \omega.$$

Show that F descends to a map $\tilde{F}$ in (deRham) cohomology H^*_{dR}, i.e. the map

$$\tilde{F} : H^p_{\mathrm{dR}}(M) \times H^q_{\mathrm{dR}}(M) \to H^{p+q}_{\mathrm{dR}}(M), \quad ([\eta], [\omega]) \mapsto [\eta \wedge \omega]$$

is well defined.

Here the brackets [ ] denote equivalence class. If $\eta \in \mathcal{A}^p(M)$, then $[\eta]$ is the equivalence class of η in the quotient space $H^p_{\mathrm{dR}}(M)$, etc.