Geometry/Topology Qualifying Exam
August 2009

1. Evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{x}{1+x^4} e^{-ipx}\,dx, \quad p > 0.$$

[The emphasis in this problem is on the calculation; you do not need to justify the steps in your calculation].

2. Let $S = \{x^2 + y^2 + z^2 = 9\}$, $H = \{x^2 + y^2 = z^2 + 1\}$, and $X = S \cap H$.

(a) Is $X$ is compact? Is $X$ connected? What is $\pi_1(X, x_0)$, $x_0 \in X$?

(b) Prove that $X$ is an embedded submanifold of $\mathbb{R}^3$.

(c) Show that $y$ is a coordinate for a neighborhood of the point $(\sqrt{5}, 0, 2)$ in $X$. For the mapping

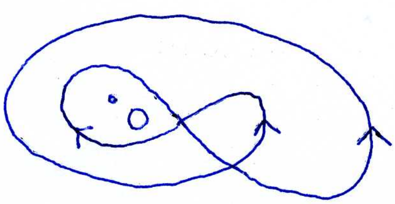
$$f : X \to \mathbb{R} : (x, y, z) \to x^2,$$

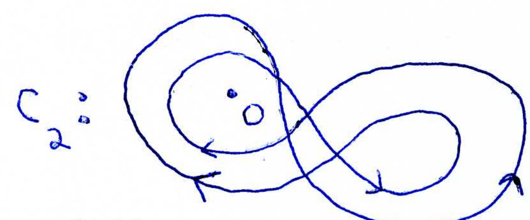
calculate $df$ in terms of the coordinate $y$.

3. Let $(r, \theta)$ denote polar coordinates for the plane $\mathbb{R}^2$ (where the coordinate neighborhood is $\mathbb{R}^2$ minus the nonnegative $x$-axis, say).

(a) Compute the form $d\theta$ in terms of Euclidean coordinates $(x, y)$, and use this to explain why $d\theta$ is actually well-defined as a one-form in $\mathbb{R}^2 \setminus \{0\}$.

(b) Compute the integrals $\int_C d\theta$ for the following oriented curves $C_1$ and $C_2$:



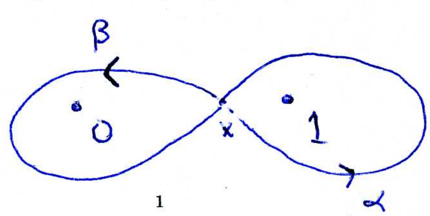

(c) Explain why $d\theta$ is not exact in all of $\mathbb{R}^2 \setminus \{0\}$.

4. (a) Explain why the map

$$f : \mathbb{C} \setminus \{0, \pm 1, \pm i\} \to \mathbb{C} \setminus \{0, 1\} : z \to w = z^4$$

is a normal (or regular) covering space.

(b) Identify the subgroup of $\pi_1(\mathbb{C} \setminus \{0, 1\}, x)$ which corresponds to the covering space in (a), where $x$ is a basepoint and $\pi_1$ is presented as in the following picture:



5. Use Mayer-Vietoris and the covering indicated below, to compute the homology of the space $T\#T$, where $T = S^1 \times S^1$:


6. For each of the following statements, either briefly explain why the statement is true, or give a counterexample.

(a) Every exact $k-$form on a compact orientable $k$-dimensional manifold vanishes at some point.

(b) If $X$ is vector field on a manifold $M$ and $X(q) \neq 0$, then there exists a coordinate system $x_1, .., x_n$ near $q$ such that $X = \frac{\partial}{\partial x_1}$.

(c) If $X$ and $Y$ are vector fields on a manifold $M$ and $X(q)$ and $Y(q)$ are independent, then there exists a coordinate system $x_1, .., x_n$ near $q$ such that $X = \frac{\partial}{\partial x_1}$ and $Y = \frac{\partial}{\partial x_2}$.

(d) There exists a compact two-manifold $X$ with $H_1(X, \mathbb{Z}) \neq 0$ and $H^1_{DR}(X, \mathbb{R}) = 0$, where $H^*_{DR}$ denotes DeRham cohomology;

(e) For a compact two-manifold $X$, if $H_1(X) = 0$, then $\pi_1(X, x_0) = 0$.