Geometry-Topology Qualifying Exam: August 2010

1. Identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, $(x, y) \simeq x + iy$. Suppose $f$ is a smooth map from $\mathbb{R}^2 \simeq \mathbb{C}$ to $\mathbb{C}$ and write $f = f_1 + if_2$ where $f_j$ are real valued functions for $j = 1, 2$.

(a) Define, $d f = df_1 + idf_2$ where $d$ is the usual exterior derivative. Then it is a straightforward calculation (which you can assume without proof) that for $z = x + iy$,

$$df = \partial f \, dz + \bar{\partial} f \, d\bar{z},$$

where,

$$\partial = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad \text{and} \quad \bar{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Use (1) to show that if $f: \mathbb{C} \rightarrow \mathbb{C}$ is an analytic function,

$$df = f'(z) \, dz,$$

where $f'(z)$ is the complex derivative of $f$.

(b) Use the real inverse function theorem to deduce that if $f: \mathbb{C} \rightarrow \mathbb{C}$ is analytic in a neighborhood of $z_0$ and $f'(z_0) \neq 0$ then $f$ is invertible in a neighborhood of $z_0$ with a smooth inverse that is also locally analytic. Hint: for the second part differentiate $f(g(z)) = z$ and solve for $dg$.

Do 5 of the following 6 problems:

2. Define a map $m: S^1 \times S^1 \rightarrow S^1$ by,

$$S^1 \times S^1 \ni (z_1, z_2) \mapsto m(z_1, z_2) = z_1z_2 \in S^1,$$

where the multiplication in $S^1$ comes from thinking of it as a subset of $\mathbb{C}$.

Define a map $\pi: S^1 \times S^1 \rightarrow S^1$ by,

$$S^1 \times S^1 \ni (z_1, z_2) \mapsto \pi(z_1, z_2) = z_1 \in S^1$$

Use degree theory to show that $m$ and $\pi$ cannot be homotopic. Hint: consider composition with $z_2 \rightarrow (1, z_2)$.

3. The parametrization $x(t) = \cosh t$, $y(t) = \sinh t$ defines a (global) coordinate $t$ on the right hand sheet, $M$, of the hyperbola $x^2 - y^2 = 1$. Define a one form $\omega$ on $M$ by contracting the volume form $dx \wedge dy$ on $\mathbb{R}^2$ with the unit normal field $n$ on $M$ (there are two such normal fields on $M$ – you can just choose one of them). That is,

$$(dx \wedge dy)_p(n, \cdot) = \omega_p(\cdot),$$

where the argument “. ” on both sides is restricted to the tangent space $T_pM$ for $p \in M$. Find $f(t)$ so that,

$$\omega = f(t) \, dt$$ on the tangent space $T_{(x(t), y(t))}M$.

4. Suppose that $p_j \in S^1$ for $j = 1, 2$. Define,

$$U = (S^1 \setminus \{p_1\}) \times S^1,$$

$$V = S^1 \times (S^1 \setminus \{p_2\}).$$
Use the Mayer-Vietoris sequence and your knowledge of the deRham cohomology of $S^1$ to calculate the deRham cohomology of the punctured torus, $S^1 \times S^1 \setminus \{(p_1, p_2)\}$.

5. Use the Mayer-Vietoris sequence and your knowledge of the singular homology of $S^1$ to calculate the singular homology of the figure eight,

\[ \bigcirc \bigcirc \] : the figure eight considered as a subset of $\mathbb{R}^2$

6. Consider the Klein bottle, $K$, obtained in the usual way by appropriately identifying opposite edges of a rectangle,

\[ \begin{array}{c}
\text{B} \\
\text{A} \\
\text{A} \\
\text{B} \\
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Use Seifert-Van Kampen theorem to determine a presentation of the fundamental group, $\pi_1(K, p)$ in terms of generators and relations. Here $p$ is any conveniently chosen point in $K$. Use this to find the first homology group of $K$ with integer coefficients.

7. Show that any continuous map $f: \mathbb{RP}^2 \to S^1$ is homotopic to a constant map. Hint: consider the induced map on the fundamental groups.