Geometry-Topology Qualifying Exam: August 2010

1. Identify $\boldsymbol{R}^2$ with $\boldsymbol{C}$ in the usual way, $(x, y) \simeq x + iy$. Suppose f is a smooth map from $\boldsymbol{R}^2 \simeq \boldsymbol{C}$ to $\boldsymbol{C}$ and write $f = f_1 + i f_2$ where f_j are real valued functions for $j = 1, 2$.

(a) Define, $df = df_1 + i\,df_2$ where d is the usual exterior derivative. Then it is a straightforward calculation (which you can assume without proof) that for $z = x + iy$,

$$df = \partial f dz + \bar{\partial} f d\bar{z}\ , \tag{1}$$

where,

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Use (1) to show that if $f\colon \boldsymbol{C} \to \boldsymbol{C}$ is an analytic function,

$$df = f'(z)\,dz,$$

where $f'(z)$ is the complex derivative of f.

(b) Use the *real* inverse function theorem to deduce that if $f\colon \boldsymbol{C} \to \boldsymbol{C}$ is *analytic* in a neighborhood of z_0 and $f'(z_0) \neq 0$ then f is invertible in a neighborhood of z_0 with a smooth inverse that is also locally analytic. Hint: for the second part differentiate $f(g(z)) = z$ and solve for dg.

Do 5 of the following 6 problems:

2. Define a map $m\colon S^1 \times S^1 \to S^1$ by,

$$S^1 \times S^1 \ni (z_1, z_2) \to m(z_1, z_2) = z_1 z_2 \in S^1,$$

where the multiplication in S^1 comes from thinking of it as a subset of $\boldsymbol{C}$.

Define a map $\pi\colon S^1 \times S^1 \to S^1$ by,

$$S^1 \times S^1 \ni (z_1, z_2) \to \pi(z_1, z_2) = z_1 \in S^1$$

Use degree theory to show that m and π cannot be homotopic. Hint: consider composition with $z_2 \to (1, z_2)$.

3. The parametrization $x(t) = \cosh t$, $y(t) = \sinh t$ defines a (global) *coordinate* t on the right hand sheet, M, of the hyperbola $x^2 - y^2 = 1$. Define a one form ω on M by contracting the volume form $dx \wedge dy$ on $\boldsymbol{R}^2$ with the unit normal field $\boldsymbol{n}$ on M (there are two such normal fields on M – you can just choose one of them). That is,

$$(dx \wedge dy)_p(\boldsymbol{n}, \cdot\,) = \omega_p(\,\cdot\,),$$

where the argument " $\cdot$ " on both sides is restricted to the tangent space T_pM for $p \in M$. Find $f(t)$ so that,

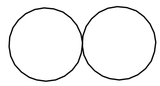
$$\omega = f(t)\,dt \text{ on the tangent space } T_{(x(t), y(t))}M.$$

4. Suppose that $p_j \in S^1$ for $j = 1, 2$. Define,

$$U = (S^1 \backslash \{p_1\}) \times S^1,$$
$$V = S^1 \times (S^1 \backslash \{p_2\}).$$

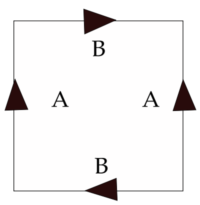
Use the Mayer-Vietoris sequence and your knowledge of the deRham cohomology of S^1 to calculate the deRham cohomology of the punctured torus, $S^1 \times S^1 \backslash \{(p_1, p_2)\}$.

5. Use the Mayer-Vietoris sequence and your knowledge of the singular homology of S^1 to calculate the singular homology of the figure eight,



: the figure eight considered as a subset of $\boldsymbol{R}^2$

6. Consider the Klein bottle, K, obtained in the usual way by appropriately identfying opposite edges of a rectangle,



Use Seifert-Van Kampen theorem to determine a presentation of the fundamental group, $\pi_1(K,\ p)$ in terms of generators and relations. Here p is any conveniently chosen point in K. Use this to find the first homology group of K with integer coefficients.

7. Show that any continuous map $f\colon \boldsymbol{RP}^2 \to S^1$ is homotopic to a constant map. Hint: consider the induced map on the fundamental groups.