Geometry/Topology Qualifying Exam

August 2011

1. Compute the following integral:

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{b^2 + x^2} dx,$$

where $b > 0$ and $\omega > 0$, using an appropriate contour integral.

2. a) Show that every continuous map $S^2 \to S^1 \times S^1$ is nullhomotopic (i.e., homotopic to a constant map).

b) Show that the map $S^1 \times S^1 \to S^2$ gotten by collapsing two generating curves to a point is not nullhomotopic. (You can picture this map by considering the torus $S^1 \times S^1$ as the square with the appropriate identifications on the boundary and then S^2 as the square with the entire boundary identified to one point.)

3. For the following forms, show each is closed, exact, both, or neither:

a) $\omega_1 = xdy - ydx$ on $\mathbb{R}^2$

b) $\omega_2 = \frac{xdy - ydx}{x^2+y^2}$ on $\mathbb{R}^2 \setminus \{(0,0)\}$.

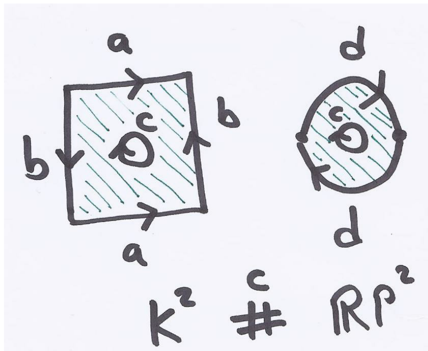
c) $\iota^*\omega_1$, where ι is the inclusion of the circle of radius 1 centered at $(0,0)$ into $\mathbb{R}^2$.

d) $\iota^*\omega_2$, where ι is the inclusion of the circle of radius 1 centered at $(0,0)$ into $\mathbb{R}^2$.

Note: in c and d, we mean the restriction of the forms to the unit circle, and are considering the forms on the circle.

Do three of the following four problems:

4. Give a presentation of the fundamental group of the connected sum $K^2 \# \mathbb{RP}^2$ constructed as shown (with gluings along a, b, c, d):



where the basepoint is on the curve c.

5. Let $a : S^n \to S^n$ be the antipodal map $a(x) = -x$ on the n-sphere S^n.
a) Show that a is orientation preserving if and only if n is odd.
b) Show that $\mathbb{RP}^n$ is orientable if and only if n is odd.

6. Consider the space $\mathbb{R}^3 \setminus C$, where C is the union of the x and y axes.
a) Explain why the homology groups with integer coefficients of $\mathbb{R}^3 \setminus C$ are isomorphic to the homology groups with integer coefficients of S^2 minus four points.
b) Compute the homology with integer coefficients of $\mathbb{R}^3 \setminus C$.

7. Let M be a smooth, closed (compact), orientable n-dimensional manifold, let $p \in M$ and let $B \subseteq M$ be an open coordinate ball containing p (so B is diffeomorphic to a ball in $\mathbb{R}^n$). Let $A = M \setminus \{p\}$. Using Mayer-Vietoris for de Rham cohomology with the cover $M = A \cup B$, show:
a) The connecting homomorphism $H^{n-1}_{dR}(A \cap B) \to H^n_{dR}(M)$ is an isomorphism.
b) The map $H^{n-1}_{dR}(M) \to H^{n-1}_{dR}(A)$ induced by the inclusion map is an isomorphism.