Geometry/Topology Qualifying Exam

August 2011

1. Compute the following integral:

\[ \int_{-\infty}^{\infty} \frac{\cos \omega x}{b^2 + x^2} \, dx, \]

where \( b > 0 \) and \( \omega > 0 \), using an appropriate contour integral.

2. a) Show that every continuous map \( S^2 \to S^1 \times S^1 \) is nullhomotopic (i.e., homotopic to a constant map).

b) Show that the map \( S^1 \times S^1 \to S^2 \) gotten by collapsing two generating curves to a point is not nullhomotopic. (You can picture this map by considering the torus \( S^1 \times S^1 \) as the square with the appropriate identifications on the boundary and then \( S^2 \) as the square with the entire boundary identified to one point.)

3. For the following forms, show each is closed, exact, both, or neither:

a) \( \omega_1 = xdy - ydx \) on \( \mathbb{R}^2 \)

b) \( \omega_2 = \frac{x\,dy - y\,dx}{x^2 + y^2} \) on \( \mathbb{R}^2 \setminus \{(0,0)\} \).

c) \( \iota^* \omega_1 \), where \( \iota \) is the inclusion of the circle of radius 1 centered at \( (0,0) \) into \( \mathbb{R}^2 \).

d) \( \iota^* \omega_2 \), where \( \iota \) is the inclusion of the circle of radius 1 centered at \( (0,0) \) into \( \mathbb{R}^2 \).

Note: in c and d, we mean the restriction of the forms to the unit circle, and are considering the forms on the circle.
Do three of the following four problems:

4. Give a presentation of the fundamental group of the connected sum $K^2 \# \mathbb{RP}^2$ constructed as shown (with gluings along $a$, $b$, $c$, $d$):

![Diagram of $K^2 \# \mathbb{RP}^2$]

where the basepoint is on the curve $c$.

5. Let $a : S^n \to S^n$ be the antipodal map $a(x) = -x$ on the $n$-sphere $S^n$.
   a) Show that $a$ is orientation preserving if and only if $n$ is odd.
   b) Show that $\mathbb{RP}^n$ is orientable if and only if $n$ is odd.

6. Consider the space $\mathbb{R}^3 \setminus C$, where $C$ is the union of the $x$ and $y$ axes.
   a) Explain why the homology groups with integer coefficients of $\mathbb{R}^3 \setminus C$ are isomorphic to the homology groups with integer coefficients of $S^2$ minus four points.
   b) Compute the homology with integer coefficients of $\mathbb{R}^3 \setminus C$.

7. Let $M$ be a smooth, closed (compact), orientable $n$-dimensional manifold, let $p \in M$ and let $B \subseteq M$ be an open coordinate ball containing $p$ (so $B$ is diffeomorphic to a ball in $\mathbb{R}^n$). Let $A = M \setminus \{p\}$. Using Mayer-Vietoris for de Rham cohomology with the cover $M = A \cup B$, show:
   a) The connecting homomorphism $H^{n-1}_{dR}(A \cap B) \to H^n_{dR}(M)$ is an isomorphism.
   b) The map $H^{n-1}_{dR}(M) \to H^{n-1}_{dR}(A)$ induced by the inclusion map is an isomorphism.