Geometry and Topology Qualifying Exam, August 2012

1. Find curvature of the curve in $\mathbf{R^3}$, defined parametrically by

$$r(t) = (x(t), y(t), z(t) = (t^2, 1-t, t^3).$$

State the definition of the torsion of a space curve, such as $r(t)$ (you are not asked to calculate it).

2. Evaluate the *Fresnel integral*

$$\lim_{R\to\infty} \int_0^R \sin(x^2)\, dx.$$

Hint: What is e^{iz^2} for $\arg z = \frac{\pi}{4}$?

3. Let M be a $2n$-dimensional manifold and ω—a differential 2-form on M. Suppose around each point of M we can find local coordinates p_j, q_j, $j = 1, \dots n$, in which ω equals $\sum_{j=1}^n dp_j \wedge dq_j$.

a) Show that ω is closed.

b) Show that if $M = \mathbf{R}^{2n}$, with coordinates p_j, q_j, $j = 1, \dots n$ then ω defined globally as $\sum_{j=1}^n dp_j \wedge dq_j$ is exact.

4. Compute the homology groups with coefficients in $\mathbf{Z}$ of one of the following spaces:

a) $\mathbf{RP}^2 \vee \mathbf{T}^2$, where $\mathbf{T}^2$ is the two-dimensional torus and $X \vee Y$ denotes the wedge of two spaces, i.e. the quotient space of the disjoint union of X and Y by the relation which identifies a point $x_0 \in X$ with a point $y_0 \in Y$.

or

b) the space obtained by pinching one of the circles of $\mathbf{T}^2$ to a point. That is, if $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$, we identify all points (z_0, w) for a fixed z_0 and consider the quotient space of $\mathbf{T}^2$ by this equivalence relation.

5. a) Find the degree of the map $f : \mathbf{S}^1 \to \mathbf{S}^1$ defined by $f(z) = z^k$, $k \in \mathbf{Z}$, where $\mathbf{S}^1 = \{z \in \mathbf{C} | |z| = 1\}$ is the unit circle with the standard orientation. Please state the definition and the properties of the degree of a map you are using.

b) Show that for two smooth maps $f, g : \mathbf{S}^1 \to \mathbf{S}^1$, $\deg(f \circ g) = \deg f \cdot \deg g$.

c) Use results of parts a) and b) to construct a monomorphism—i.e. an injective homomorphism—of $\mathbf{Z}$ into the fundamental group of $\mathbf{S}^1$ (actually, an isomorphism, but you are not asked to prove it here).

6. Are the following statements true or false? Give short justifications or counterexamples.

a) An annulus $A = \{z \in \mathbf{C} | 1 < |z| < 2\}$ has a nontrivial first homology group, because there exist a 1-cycle c and a smooth 1-form ω in A, such that $\int_c \omega \neq 0$.

b) For a simply connected space X, $H_1(X; \mathbf{Z}) = 0$ in the sense of singular homology.

c) There exists a covering $p : \mathbf{T}^2 \to \mathbf{S}^2$.

d) There is no smooth closed 2-form on $\mathbf{R}^3$ whose restriction to S^2 would be dA—the area form inherited from $\mathbf{R}^3$

e) Let M and N be smooth manifolds of the same dimension. A smooth bijection $f : M \to N$ with nondegenerate tangent map (differential) df_x at every point $x \in M$ is a diffeomorphism.

GOOD LUCK!