Geometry/Topology Qualifying Exam

August 2017

Please show all your work. GOOD LUCK!

Problem 1

Evaluate

$$\int_0^\infty \frac{dx}{1+x^6}.$$

Problem 2

Consider the function $F : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$F(x, y, z, w) = (x^2 - y^2, wz).$$

a) Show that $M = F^{-1}(1, 1)$ is a manifold and compute its dimension.
b) Give a basis for the tangent space of M (viewed as a subspace of $\mathbb{R}^4$) at $(1, 0, 1, 1)$.
c) Let $S = F^{-1}(0, 0)$. Describe all points $(x, y, z, w) \in S$ for which the Implicit Function Theorem or Regular Value Theorem can be used to show (x, y, z, w) has a neighborhood in S diffeomorphic to an open set in Euclidean space.

Problem 3

Consider the following vector fields and forms on $\mathbb{R}^n$:

$$X = \sum_{j=1}^{n} x^j \frac{\partial}{\partial x^j},$$

$$\omega = dx^1 \wedge \cdots \wedge dx^n,$$

$$\theta = \sum_{j=1}^{n} (-1)^{j+1} x^j dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n,$$

where the hat means that term is missing. Find the following:
a) $d\theta$, where d is the exterior derivative.
b) The Lie derivative $L_X\omega$.
c) The Lie derivative $L_X\theta$.

Problem 4

Let S^n be a sphere of dimension n.
a) Prove that for odd-dimensional spheres, the identity mapping $S^{2n+1} \to S^{2n+1}$ is homotopic to the antipodal map $a(x) = -x$.
Hint: You may realize S^{2n+1} as

$$\{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : |z_1|^2 + \cdots |z_{n+1}|^2 = 1\}.$$

b) Show that for every n, the map induced on the singular homology groups $H_k(S^n)$ by the antipodal map (for each k) is an isomorphism.

Problem 5

The space X_n is obtained from $S^n \times [0,1]$ by identifying points $(x, 1)$, $x \in S^n$, with $(-x, 0)$. Here $-x$ is the antipodal point to x. For $n > 1$, find $\pi_1(X_n)$.

Problem 6

Let M be a compact, orientable manifold without boundary of dimension n. Prove that $H^n_{DR}(M) \neq 0$. Here H_{DR} are De Rham cohomology groups.