Geometry-Topology qualifying exam: January 2007

December 5, 2006

1. A proper rational function, $R(z)$, is a ratio $R(z) = \frac{p(z)}{q(z)}$, with $p(z)$ and $q(z)$ polynomials such that the degree of $p(z)$ is strictly less than the degree of $q(z)$. Use Liouville's theorem to show that any proper rational function is the sum of the principal parts at its poles. Recall that the principal part of a Laurent expansion at a pole type singularity at $z = a$ is the sum of the terms of the form $c_n(z-a)^n$ for $n = -1, -2, -3\ldots$

2. The exponential map, $\exp : \mathbf{C} \to \mathbf{C}^*$, is a covering space map. Suppose that X is a path connected smooth manifold and that $\varphi : X \to \mathbf{C}^*$ is a smooth map. Let $z = x + iy$ denote the usual complex coordinate on $\mathbf{C}$.

(a) Show that the one form ($\Re$=real part),

$$\eta := \Re\left(\frac{dz}{2\pi i z}\right),$$

is a generator for the de Rham cohomology $H^1(\mathbf{C}^*)$.

(b) Show that if $\varphi^*(\eta) = 0$ in $H^1(X)$ then there exists a smooth lift $\tilde{\varphi} : X \to \mathbf{C}$ such that,

$$\varphi = \exp\tilde{\varphi}.$$

3. Let D be the closed unit disk $\{(x, y) \in \mathbf{R}^2 : \ x^2 + y^2 \leq 1\}$ and let X denote the topological space obtained from D by identifying antipodal points $(x, y) \simeq -(x, y)$ on the *boundary* $(x^2 + y^2 = 1)$ of D. Explain the use of the Siefert-van Kampen theorem to calculate the fundamental group, $\pi_1(X, p)$ for some point $p \in X$. Identify a path that is a generator of $\pi_1(X, p)$.

4. Let M be a compact connected orientable manifold with dimension $n \geq 2$. Suppose that p is a point in M and V is a neighborhood of p which is diffeomorphic to $\mathbf{R}^n$. Let $U = M \setminus \{p\}$. Show that the connecting homomorphism $d^* : H^{n-1}(U \cap V) \to H^n(M)$ in the Mayer-Vietoris sequence for de Rham cohomology is an isomorphism.

5. Let $d\theta$ denote the one form on S^1 that is the derivative of some local choice of angle $(x, y) = (\cos\theta, \sin\theta) \in S^1$. Let $S^1 \ni (x, y) \to p = \frac{x}{1-y}$ denote stereographic projection from the north pole. This is a coordinate defined for $y \neq 1$. Find a function $f(p)$ so that,

$$d\theta = f(p)dp.$$

6. Show that the subset, M, of $\mathbf{R}^3$ defined by the equation,

$$(1 - z^2)(x^2 + y^2) = 1,$$

is a smooth submanifold of $\mathbf{R}^3$. Define a vector field on $\mathbf{R}^3$ by,

$$V = z^2 x \frac{\partial}{\partial x} + z^2 y \frac{\partial}{\partial y} + z(1 - z^2)\frac{\partial}{\partial z}.$$

(a) Show that the restriction of V to M is a tangent vector field to M.

(b) The map family of maps $\varphi_t(x, y, z) = (cx - sy, sx + cy, z)$ with $c = \cos t$ and $s = \sin t$ obviously restricts to a one parameter family of diffeomorphisms of M. For each t determine the vector field $(\varphi_t)_* V$ on M.

7. Find the critical points and critical values for the function $f(x, y, z) = x^2 + y^2 - z$ restricted to the two sphere $x^2 + y^2 + z^2 = 1$.