GEOMETRY-TOPOLOGY EXAM. JANUARY 2008.

1. Find a conformal mapping, $w = f(z)$, which maps the annulus $1 < |z| < 2$ minus the line segment $\{z \mid -2 < \mathrm{Re}(z) < -1, \mathrm{Im}(z) = 0\}$ onto the open unit square $0 < \mathrm{Re}(w) < 1$, $0 < \mathrm{Im}(w) < 1$.

2. Consider the map $f : \mathbb{R}^4 \to \mathbb{R}$ given by $f(x, y, z, t) = x^2 + y^2 - z^2 - t^2$ restricted to the unit 3-sphere $\mathbb{S}^3 \hookrightarrow \mathbb{R}^4$. For which values $a \in \mathbb{R}$ is the level set $f^{-1}(a) \cap \mathbb{S}^3$ a smooth embedded submanifold of $\mathbb{S}^3$?

3. Consider the torus $\mathbb{T}^2 \subset \mathbb{R}^3$ parameterized as $x = (R+r\cos\theta)\cos\psi$, $y = (R+r\cos\theta)\sin\psi$, $z = r\sin\theta$, where $0 < r < R$ are given constants. Compute the area form on $\mathbb{T}^2$ induced by the standard volume form $dx \wedge dy \wedge dz$ on $\mathbb{R}^3$.

4. Let $M = \mathbb{RP}^2$ with homogeneous coordinates $[y_0 : y_1 : y_2]$. Consider the chart U_0 given by $y_0 \neq 0$ with coordinates $x_1 = y_1/y_0$ and $x_2 = y_2/y_0$. Let X be the vector field given by the formula $X = x_2 \dfrac{\partial}{\partial x_1}$. Compute this vector field in the charts $\{U_1 : y_1 \neq 0\}$ and $\{U_2 : y_2 \neq 0\}$ and determine whether X extends smoothly to the whole of M.

5. Let K be the Klein bottle, whose fundamental group $\pi_1(K, x)$ is generated in the standard way by elements a and b with a single relation $abab^{-1} = 1$. (See Figure 1.) Describe the covering space X of K corresponding to the subgroup of $\pi_1(K, x)$ generated by a and b^2, determine whether it is normal and describe the group of covering automorphisms of X.

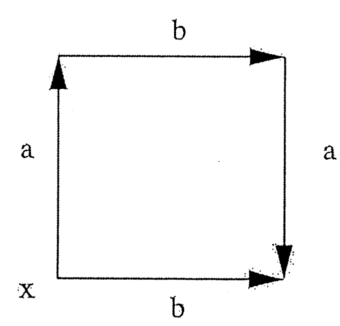

FIGURE 1

6. Let F be the space of ordered pairs of orthonormal vectors in $\mathbb{R}^3$ with the natural topology. I. e. a point in F is a pair of vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $|\mathbf{v}| = |\mathbf{w}| = 1$ and $\mathbf{v} \cdot \mathbf{w} = 0$. Find a suitable cellular decomposition of F and compute its homology.

7. Prove that every continuous map $h : \mathbb{RP}^2 \to \mathbb{RP}^2$ has a fixed point.