Geometry/Topology Qualifying Exam
January 2009

1. Calculate
\[ \int_0^\infty \frac{2x^2 - 1}{x^4 + 5x^2 + 4} \, dx. \]

2. Consider the function
\[ f : S^2 \to \mathbb{R} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to xy. \]
Find the critical points and critical values for this function.

3. Consider the following vector fields defined in \( \mathbb{R}^2 \):
\[ X = 2 \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \quad \text{and} \quad Y = \frac{\partial}{\partial y}. \]
Determine whether or not there exists a (locally defined) coordinate system \((s,t)\) in a neighborhood of \((x,y) = (0,1)\) such that
\[ X = \frac{\partial}{\partial s}, \quad \text{and} \quad Y = \frac{\partial}{\partial t}. \]

4. Let \( X \) be a connected manifold and let \( S^1 \) be the unit circle. Recall that \( X \vee S^1 \) is the space obtained by identifying one point in \( X \) with one point in \( S^1 \). Determine whether the \( S^1 \) is homotopically trivial in the space \( X \vee S^1 \).

5. The 3-ball \( B^3(r) \subseteq \mathbb{R}^3 \) is a 3-manifold with boundary \( S^2(r) \), the 2-sphere of radius \( r \). Equip \( B^3(r) \) with the standard orientation and \( S^2(r) \) with the induced orientation. Assume that \( \omega \) is a 2-form defined on \( \mathbb{R}^3 \setminus \{0\} \) such that
\[ \int_{S^2(r)} \omega = a \frac{b}{r}, \]
for all \( r > 0 \).

(a) Given \( 0 < c < d \), let \( M = \{ x \in \mathbb{R}^3 : c \leq |x| \leq d \} \), with standard orientation. Evaluate \( \int_M \omega \).

(b) If \( \omega \) is closed, what can you say about \( a \) and \( b \)?

(c) If \( \omega \) is exact in \( \mathbb{R}^3 \setminus \{0\} \), what can you say about \( a \) and \( b \)?

6. Let \( \Gamma \) denote the group generated by the transformations of \( \mathbb{R}^2 \) given by
\[ A : (x,y) \to (x+1,-y) \]
and
\[ B : (x,y) \to (x,y+1). \]

(a) Identify the surface \( M \) obtained from \( \mathbb{R}^2 \) by identifying \( (x,y) \) and \( \gamma(x,y) \), for each \( \gamma \in \Gamma \).

(b) Find explicit generators for the DeRham cohomology of the surface \( M \) (using the variables \( x \) and \( y \)).
7. Determine whether each of the following statements is true or false, and briefly explain.

(a) The tangent bundle of $S^2$ is a trivial vector bundle.
(b) The tangent bundle of $S^3$ is a trivial vector bundle.
(c) The universal covering space of $\mathbb{R}^2 \setminus \{\pm 1\}$ is contractible.
(d) If the degree of a smooth map $f : S^2 \to S^2$ is nonzero, then the map $f$ is onto.
(e) All covering spaces of the torus $S^1 \times S^1$ are normal.