1. Compute the following integral:
\[ \int_{-\infty}^{\infty} \frac{x^2}{a^4 + x^4} \, dx, \]
where \( a > 0 \) using an appropriate contour integral.

2. Suppose \( f : M \to \mathbb{R} \) is a smooth function. Consider the local definition of \( df \) by
\[ df = \frac{\partial f}{\partial x^i} \, dx^i \]
in coordinate \( (x^1, \ldots, x^n) \) on a neighborhood of a point \( p \in M \), where we have used the Einstein summation convention.

a) Show that \( df \) is a well-defined global object by proving that if \( (y^1, \ldots, y^n) \) is another coordinate in a neighborhood of the point \( p \), then
\[ \frac{\partial f}{\partial x^i} \, dx^i = \frac{\partial f}{\partial y^j} \, dy^j \]
in a neighborhood of \( p \). (Hint: you will need the transition maps going from \( (x^1, \ldots, x^n) \) to \( (y^1, \ldots, y^n) \) in order to do this calculation.)

b) Show that the naive definition of the Hessian (second derivative) given locally by
\[ \frac{\partial^2 f}{\partial x^i \partial x^j} \, dx^i \otimes dx^j \]
in coordinate \( (x^1, \ldots, x^n) \) on a neighborhood of a point \( p \in M \) does not, in general, give a well-defined global object (i.e., it does not satisfy an equality as above when one changes coordinates). In addition, show that it is well-defined at a point \( p \) where \( df_p = 0 \).
3. For this problem, let \( X \) be a simply-connected topological space.

a) Show that any continuous map \( \phi : X \to S^2 \) that is not surjective is homotopic to a constant map. In the special case \( X = S^1 \), find an explicit homotopy between a constant map and the mapping \( \psi : S^1 \to S^2 \) that maps the circle to the equator of \( S^2 \) in \( \mathbb{R}^3 \), i.e., \( \psi(x, y) = (x, y, 0) \).

b) Show that any continuous map \( f : X \to S^1 \) is homotopic to a constant map. In the special case \( X = S^1 \), find an explicit homotopy between a constant map and the mapping \( \phi : S^1 \to S^2 \) that maps the circle to the equator of \( S^2 \) in \( \mathbb{R}^3 \); i.e., \( (x; y) = (x; y; 0) \).

4. a) Use Van Kampen’s theorem to show that the fundamental group of the torus \( S^1 \times S^1 \) is presented by \( \langle a, b : ab = ba \rangle \) using the model of the torus given by a square with opposite sides identified. Take the basepoint to be near the boundary of the square.

b) Recall that the surface \( \Sigma_g \) of genus \( g \) can be obtained by taking the connected sum of \( g \) tori. Show the surface of genus two has fundamental group presented by \( \langle a_1, b_1, a_2, b_2 : a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1} = 1 \rangle \) by using Van Kampen’s theorem and decomposing the \( \Sigma_2 \) as a connected sum of two tori. Be sure to choose an appropriate basepoint.

5. a) Suppose \( \omega \) is a smooth, exact \( k \)-form. Show that \( \omega \wedge \omega \) is an exact \((2k)\)-form.

b) Suppose \( \omega \) is a smooth, closed \( 2 \)-form on \( S^4 \). Show that \( \omega \wedge \omega \) vanishes somewhere.

6. The suspension \( \Sigma X \) of a space \( X \) is defined by taking the space \( X \times [0, 1] \) and identifying all the points of \( X \times \{0\} \) together and all the points of \( X \times \{1\} \) together. For instance, the suspension of the circle consists of two cones joined at their base circles (homeomorphic to the 2-sphere).

Let \( \pi : X \times [0, 1] \to \Sigma X \) be the quotient projection. Here are some facts about \( \Sigma X \) that you may use in the following problem:

A) \( \pi(X \times \{1/2\}) \) is homeomorphic to \( X \).

B) \( \pi(X \times \{0, 1\}) \) is homeomorphic to \( X \times (0, 1) \), where \( (0, 1) \) is the open interval.

C) Let \( X_0 \) be the point in \( \Sigma X \) corresponding to \( X \times \{0\} \) and \( X_1 \) be the point in \( \Sigma X \) corresponding to \( X \times \{1\} \), i.e., \( \pi(X \times \{0\}) = \{X_0\} \) and \( \pi(X \times \{1\}) = \{X_1\} \). Then \( \Sigma X \setminus \{X_0\} \) and \( \Sigma X \setminus \{X_1\} \) are homotopy equivalent to the single point spaces \( \{X_1\} \) and \( \{X_0\} \) respectively.

Use the Mayer-Vietoris sequence to show that the reduced homology groups satisfy \( H_{i+1}(\Sigma X) \cong H_i(X) \) for all \( i \geq 0 \) and \( H_0(\Sigma X) = 0 \). (If you are not familiar with reduced homology, use Mayer-Vietoris to compute the homology groups of \( \Sigma X \) in terms of the homology groups of \( X \).)